

SLOW ENTROPY AND SYMPLECTOMORPHISMS OF COTANGENT BUNDLES

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ABSTRACT. We consider an entropy-type invariant which measures the polynomial volume growth of submanifolds under the iterates of a map, and we establish sharp uniform lower bounds of this invariant for the following classes of symplectomorphisms of cotangent bundles over a compact base:

- non-identical compactly supported symplectomorphisms which are symplectically isotopic to the identity,
- symplectomorphisms generated by classical Hamiltonian functions,
- Dehn twist like symplectomorphisms over compact rank one symmetric spaces.

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1. INTRODUCTION AND MAIN RESULTS

The topological entropy $h_{\text{top}}(\varphi)$ of a compactly supported smooth diffeomorphism φ of a smooth manifold M is a basic numerical invariant measuring the orbit structure complexity of φ . There are various ways of defining $h_{\text{top}}(\varphi)$, see [29]. A geometric way was found by Yomdin and Newhouse in their seminal works [61] and [35]: Fix a Riemannian metric g on M . For $i \in \{1, \dots, \dim M\}$ denote by Σ_i the set of smooth embeddings σ of the cube $Q^i = [0, 1]^i$ into M , and by $\mu_g(\sigma)$ the volume of $\sigma(Q^i) \subset M$ computed with respect to the measure on $\sigma(Q^i)$ induced by g . The *i-dimensional volume growth* $v_i(\varphi)$ of φ is defined as

$$v_i(\varphi) = \sup_{\sigma \in \Sigma_i} \liminf_{n \rightarrow \infty} \frac{\log \mu_g(\varphi^n(\sigma))}{n}.$$

Since φ is compactly supported, $v_i(\varphi)$ does not depend on the choice of g and is finite, and $v_{\dim M}(\varphi) = 0$. The main result of [61, 35] is

$$(1) \quad \max_i v_i(\varphi) = h_{\text{top}}(\varphi).$$

The purpose of this work is to study the volume growth of symplectomorphisms of cotangent bundles T^*B over a closed base B endowed with their canonical symplectic structure $\omega = d\lambda$. Cotangent bundles are the phase spaces of classical mechanics, and classical Hamiltonian systems on such manifolds describe systems without friction. The orbit structure of their time-1-maps is therefore often intricate, and so one can expect that there exist non-trivial lower bounds of entropy-type invariants for non-identical Hamiltonian diffeomorphisms on cotangent bundles.

The topological entropy itself, which by (1) measures the exponential volume growth, vanishes for many non-identical Hamiltonian diffeomorphisms. Following [30], we therefore look at the polynomial range of growth and define for each compactly supported symplectomorphisms of T^*B and each $i \in \{1, \dots, 2d = \dim T^*B\}$ the *i-dimensional slow volume growth* $s_i(\varphi) \in [0, \infty]$ by

$$(2) \quad s_i(\varphi) = \sup_{\sigma \in \Sigma_i} \liminf_{n \rightarrow \infty} \frac{\log \mu_g(\varphi^n(\sigma))}{\log n}.$$

Again, $s_i(\varphi)$ does not depend on the choice of g , and $s_{2d}(\varphi) = 0$.

We shall establish uniform lower bounds of s_1 for certain classes of symplectomorphisms which are symplectically isotopic to the identity and a uniform lower bound of s_d for certain symplectomorphisms some of which are smoothly but not symplectically isotopic to the identity. For the moment, all Hamiltonian functions and all symplectomorphisms are assumed to be C^∞ -smooth. Weaker smoothness assumptions are discussed in Section 6.

We denote by $\text{Symp}_0^c(T^*B)$ the identity component of the group $\text{Symp}^c(T^*B)$ of compactly supported symplectomorphisms of $(T^*B, d\lambda)$. It contains the group of compactly supported Hamiltonian diffeomorphisms generated by time-dependent compactly supported Hamiltonian functions $H: [0, 1] \times T^*B \rightarrow \mathbb{R}$.

Theorem 1. *For every non-identical symplectomorphism $\varphi \in \text{Symp}_0^c(T^*B)$ it holds true that $s_1(\varphi) \geq 1$.*

Theorem 1 is sharp. Indeed, we shall show by means of an example that

Proposition 1. *On every $2d$ -dimensional symplectic manifold (M, ω) there exists a compactly supported Hamiltonian diffeomorphism φ such that $s_i(\varphi) = 1$ for all $i = 1, \dots, 2d - 1$.*

Theorem 1 implies at once that the group $\text{Symp}_0^c(T^*B)$ has no torsion. Stronger implications for the algebraic structure of the groups $\text{Ham}^c(T^*B)$ and $\text{Symp}_0^c(T^*B)$ are given in Section 3, where we work with arbitrary exact symplectic manifolds convex at infinity.

We next look at classical Hamiltonian systems. We choose a Riemannian metric g on B and denote by g^* the Riemannian metric induced on T^*B . We denote canonical coordinates on T^*B by (q, p) . A classical Hamiltonian function $H: \mathbb{R} \times T^*B \rightarrow \mathbb{R}$ is of the form

$$H(t, q, p) = \frac{1}{2} |p - A(t, q)|^2 + V(t, q)$$

and periodic in the time variable t . For the purpose of this paper we can assume without loss of generality that the period is 1. Writing $S^1 = \mathbb{R}/\mathbb{Z}$ we then have $H: S^1 \times T^*B \rightarrow \mathbb{R}$. It is well-known that such Hamiltonian functions generate a flow. We denote its time-1-map by φ_H , and we use the Riemannian metric g^* to define $s_1(\varphi_H)$ by (2). Since φ_H is not compactly supported, $s_1(\varphi_H)$ depends on the choice of g . For $r > 0$ we abbreviate

$$T_r^*B = \{(q, p) \in T^*B \mid |p| \leq r\}.$$

Theorem 2. *Assume that B is a closed manifold whose fundamental group is finite or contains infinitely many conjugacy classes, and that $H: S^1 \times T^*B \rightarrow \mathbb{R}$ is a classical Hamiltonian function. Then the following assertions hold true.*

- (i) $s_1(\varphi_H) \geq 1/2$;
- (ii) $s_1(\varphi_H) \geq 1$ provided that there exists $r < \infty$ such that H is time-independent on $T^*B \setminus T_r^*B$.

Notice that the fundamental group $\pi_1(B)$ of a closed manifold B is finitely presented. As was pointed out to us by Indira Chatterji, Rostislav Grigorchuk and Guido Mislin, no infinite finitely presented group with finitely many conjugacy classes is known, and so Theorem 2 possibly holds for all closed manifolds.

Examples. We give two classes of closed manifolds B for which the assumption on $\pi_1(B)$ in Theorem 2 is met. Denote by $\mathcal{C}(B)$ the set of conjugacy classes of $\pi_1(B)$.

1. Assume that $\pi_1(B)$ is abelian. Then $\mathcal{C}(B) = \pi_1(B)$. Examples of closed manifolds with abelian fundamental group are Lie groups and, more generally, H -spaces.
2. Assume that the first Betti number $b_1(B)$ does not vanish. Then $\mathcal{C}(B)$ is infinite. Indeed, the Hurewicz map factors as

$$\pi_1(B) \rightarrow \mathcal{C}(B) \rightarrow \frac{\pi_1(B)}{[\pi_1(B), \pi_1(B)]} = H_1(B; \mathbb{Z}).$$

We in particular see that Theorem 2 applies to all closed 2-manifolds and their products. \diamond

We finally look at certain compactly supported symplectomorphisms which are not Hamiltonian. The spaces we shall consider are the cotangent bundles over compact rank one symmetric spaces (CROSS'es, for short), and the diffeomorphisms are Dehn twist like symplectomorphisms. These maps were introduced to symplectic topology by Arnol'd [3] and Seidel [46, 47]. They play a prominent role in the study of the symplectic mapping class group of various symplectic manifolds, [31, 45, 46, 47, 52], and generalized Dehn twists along spheres can be used to detect symplectically knotted Lagrangian spheres, [46, 47], and (partly through their appearance in Seidel's long exact sequence in symplectic Floer homology) are an important ingredient in attempts to prove Kontsevich's homological mirror symmetry conjecture, [31, 48, 49, 50, 53, 54].

Let (B, g) be a CROSS, i.e., B is a sphere S^d , a projective space \mathbb{RP}^d , \mathbb{CP}^n , \mathbb{HP}^n , or the exceptional symmetric space F_4/Spin_9 diffeomorphic to the Cayley plane \mathbb{CaP}^2 . All geodesics on (B, g) are embedded circles of equal length. We define ϑ to be the compactly supported diffeomorphism of T^*B whose restriction to the cotangent bundle $T^*\gamma \subset T^*B$ over any geodesic circle $\gamma \subset B$ is the square of the ordinary Dehn twist along γ depicted in Figure 1.

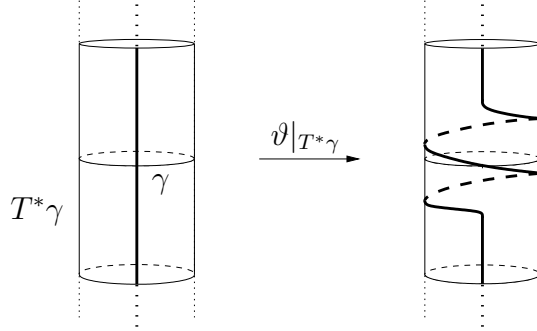


FIGURE 1. The map $\vartheta|_{T^*\gamma}$.

We call ϑ a *twist*. A more analytic description of twists is given in Subsection 5.2. It is known that twists are symplectic, and that the class of a twist generates an infinite cyclic subgroup of the mapping class group $\pi_0(\text{Symp}^c(T^*B))$, see [47].

A d -dimensional submanifold L of T^*B is called *Lagrangian* if ω vanishes on $TL \times TL$. Lagrangian submanifolds play a fundamental role in symplectic geometry. For each $\varphi \in \text{Symp}^c(T^*B)$ we therefore also consider its *Lagrangian volume growth*

$$l(\varphi) = \sup_{\sigma \in \Lambda} \liminf_{n \rightarrow \infty} \frac{\log \mu_g(\varphi^n(\sigma))}{\log n}$$

where Λ is the set smooth embeddings $\sigma: Q^d \hookrightarrow T^*B$ for which $\sigma(Q^d)$ is a Lagrangian submanifold of T^*B . Of course, $l(\varphi) \leq s_d(\varphi)$. As we shall see in Subsection 5.2, $s_i(\vartheta^m) = l(\vartheta^m) = 1$ for every $i \in \{1, \dots, 2d-1\}$ and every $m \in \mathbb{Z} \setminus \{0\}$.

Theorem 3 *Let B be a d -dimensional compact rank one symmetric space, and let ϑ be the twist of T^*B described above. Assume that $\varphi \in \text{Symp}^c(T^*B)$ is such that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(T^*B))$ for some $m \in \mathbb{Z} \setminus \{0\}$. Then $s_d(\varphi) \geq l(\varphi) \geq 1$.*

Theorem 3 is of particular interest if B is S^{2n} or \mathbb{CP}^n , $n \geq 1$, since in these cases it is known that ϑ can be deformed to the identity through

compactly supported *diffeomorphisms*, see [32, 47]. Twists can be defined on the cotangent bundle of any Riemannian manifold with periodic geodesic flow. In Section 5, Theorem 3 is proved for all known such manifolds.

In the case that B is a sphere S^d , one can use the fact that all geodesics emanating from a point meet again in the antipode to see that the twist ϑ admits a square root $\tau \in \text{Symp}^c(T^*S^d)$. For $d = 1$, τ is the ordinary Dehn twist along a circle, and for $d \geq 2$ it is the generalized Dehn twist thoroughly studied in [45, 46, 47, 52]. Given any great circle γ in S^d , the restriction of τ to $T^*\gamma \subset T^*S^d$ is the ordinary Dehn twist along γ depicted in Figure 2.

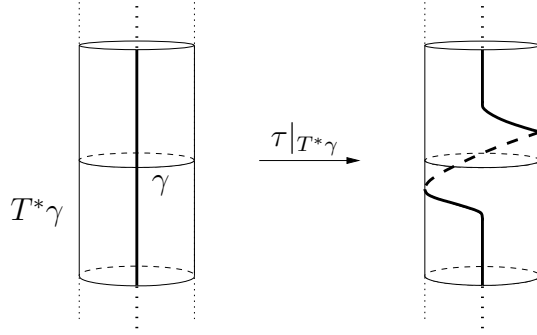


FIGURE 2. The map $\tau|_{T^*\gamma}$.

Corollary 3 *Let τ be the (generalized) Dehn twist of T^*S^d described above, and assume that $\varphi \in \text{Symp}^c(T^*S^d)$ is such that $[\varphi] = [\tau^m] \in \pi_0(\text{Symp}^c(T^*S^d))$ for some $m \in \mathbb{Z} \setminus \{0\}$. Then $s_d(\varphi) \geq l(\varphi) \geq 1$.*

Since $[\tau^2] = [\vartheta]$ has infinite order in $\pi_0(\text{Symp}^c(T^*S^d))$, so has $[\tau]$. In the case $d = 2$ it is known that $[\tau]$ generates $\pi_0(\text{Symp}^c(T^*S^2))$, see [45, 52]. Theorem 1 and Corollary 3 thus give a nontrivial uniform lower bound of the slow volume growth

$$s(\varphi) = \max_i s_i(\varphi)$$

for each $\varphi \in \text{Symp}^c(T^*S^2) \setminus \{\text{id}\}$.

Following Shub [55], we consider a symplectomorphism $\varphi \in \text{Symp}^c(T^*B)$ as a best diffeomorphism in its symplectic isotopy class if φ minimizes both $s(\varphi)$ and $l(\varphi)$. We shall show that $s_i(\tau^m) = l(\tau^m) = 1$ and $s_i(\vartheta^m) = l(\vartheta^m) = 1$ for every $i \in \{1, \dots, 2d - 1\}$ and every $m \in \mathbb{Z} \setminus \{0\}$. In view of Theorem 3 and Corollary 3, the twists τ^m and ϑ^m are then best diffeomorphisms in their symplectic isotopy classes.

Uniform lower bounds of an entropy type invariant were first obtained in a beautiful paper of Polterovich [39] for a class of symplectomorphisms of closed symplectic manifolds with vanishing second homotopy group. E.g., it is shown that for any non-identical Hamiltonian diffeomorphism φ of the standard $2d$ -dimensional torus,

$$s_1(\varphi) \geq \begin{cases} 1 & \text{if } d = 1, \\ \frac{1}{2} & \text{if } d \geq 2. \end{cases}$$

The results in [39] are formulated in terms of the growth of the uniform norm of the differential of φ . In Section 7 we shall reformulate our results in terms of this invariant.

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2. OUTLINE OF THE PROOFS

As in the previous section we consider a smooth closed manifold B and let T^*B be the cotangent bundle over B endowed with its canonical symplectic form $\omega \equiv \omega_{\text{can}} = d\lambda$ where $\lambda = \sum p_i dq_i$. We start with explaining our results for Hamiltonian diffeomorphisms. The Hamiltonian diffeomorphisms addressed in Theorem 1 are end points of smooth paths $\{\varphi^t\}$, $t \in [0, 1]$, generated by compactly supported Hamiltonian functions $H: [0, 1] \times T^*B \rightarrow \mathbb{R}$. After reparametrizing the path $\{\varphi^t\}$ we can assume that $H(t, x) = 0$ for t near 0 and t near 1, and so we can assume that H is defined on $S^1 \times T^*B$. Such a Hamiltonian function or a classical Hamiltonian function generates a flow φ_H^t . We denote the set of its 1-periodic orbits by \mathcal{P}_H . For $x \in \mathcal{P}_H$ the symplectic action is defined as

$$(3) \quad \mathcal{A}_H(x) = \int_0^1 x^* \lambda - \int_0^1 H(t, x(t)) dt.$$

Assume that there exist $x, y \in \mathcal{P}_H$ such that $\mathcal{A}_H(x) < \mathcal{A}_H(y)$. Following an idea of Polterovich [39], we shall choose a smoothly embedded

curve $\sigma: [0, 1] \rightarrow T^*B$ connecting $x(0)$ with $y(0)$ and shall show that the quantity $\int_{\varphi^n \sigma} \lambda$ grows linearly. From this we shall easily obtain

- (i) $s_1(\varphi_H) \geq 1/2$;
- (ii) $s_1(\varphi_H) \geq 1$ provided that there exists $r < \infty$ such that

$$\{\varphi^n \sigma \mid n \geq 1\} \subset T_r^*B.$$

We are then left with finding $x, y \in \mathcal{P}_H$ such that $\mathcal{A}_H(x) < \mathcal{A}_H(y)$. In the case of compactly supported Hamiltonian functions as in Theorem 1, we shall do this by using a result from [20] relying on symplectic Floer homology. In the case of the Hamiltonian functions considered in Theorem 2, we shall use work of Benci [5] to show that the symplectic action functional \mathcal{A}_H is not bounded from above on \mathcal{P}_H .

Proving Theorem 1 for the whole group $\text{Symp}_0^c(T^*B)$ is now elementary: If $\dim B \geq 2$, every $\varphi \in \text{Symp}_0^c(T^*B)$ is Hamiltonian, see the proof of Lemma 5.19 below; and for a symplectomorphism $\varphi \in \text{Symp}_0^c(T^*S^1)$ which is not Hamiltonian, the flux does not vanish, and this yields $s_1(\varphi) \geq 1$ at once.

The proof of Theorem 3 is different in nature. To fix the ideas, we assume $B = S^d$, and that φ is isotopic to ϑ through symplectomorphisms supported in $T_1^*S^d$. For $x \in S^d$ we denote by D_x the 1-disc in $T_x^*S^d$. Consider first the case $d = 1$. We fix x . For a twist ϑ as in Figure 1 and $n \geq 1$, the image $\vartheta^n(D_x)$ wraps $2n$ times around the base S^1 . For topological reasons the same must hold for φ , and so

$$\mu_{g^*}(\varphi^n(D_x)) \geq 2n\mu_g(S^1).$$

In particular, $s_1(\varphi) \geq 1$. For odd-dimensional spheres, Theorem 3 follows from a similar argument. For even-dimensional spheres, however, Theorem 3 cannot hold for topological reasons, since then ϑ is isotopic to the identity through compactly supported diffeomorphisms. In order to find a symplectic argument, we rephrase the above proof for S^1 in symplectic terms: For every $y \neq x$ the Lagrangian submanifold $\vartheta^n(D_x)$ intersects the Lagrangian submanifold D_y in $2n$ points, and under symplectic deformations of ϑ these $2n$ Lagrangian intersections persist. This symplectic point of view generalizes to even-dimensional spheres: For a twist ϑ on S^d as in Figure 1, $n \geq 1$ and $y \neq x$, the Lagrangian submanifolds $\vartheta^n(D_x)$ and D_y intersect in exactly $2n$ points. We shall prove that the Lagrangian Floer homology of $\vartheta^n(D_x)$ and D_y has rank $2n$. The isotopy invariance of Floer homology then implies that $\varphi^n(D_x)$ and D_y must intersect in at least $2n$ points. Since this holds true for every $y \neq x$, we conclude that

$$\mu_{g^*}(\varphi^n(D_x)) \geq 2n\mu_g(S^d).$$

In particular, $s_d(\varphi) \geq l(\varphi) \geq 1$.

For any compactly supported diffeomorphism φ of a manifold M we denote by

$$\rho(\varphi_*) = \lim_{n \rightarrow \infty} \|\varphi_*^n\|^{1/n}$$

the spectral radius of the induced automorphism $\varphi_*: H_*(M; \mathbb{R}) \rightarrow H_*(M; \mathbb{R})$ of the total real homology of M . It only depends on the isotopy class of φ . On his search for the simplest diffeomorphism in each isotopy class, Shub [55] formulated the entropy conjecture

$$(4) \quad h_{\text{top}}(\varphi) \geq \log \rho(\varphi_*)$$

for C^1 -diffeomorphisms. For C^∞ -diffeomorphisms the estimate (4) follows at once from Yomdin's estimate $h_{\text{top}}(\varphi) \geq \max_i v_i(\varphi)$. Besides certain Dehn twists all symplectomorphisms studied in this paper are isotopic to the identity mapping, and so the estimate (4) becomes vacuous. Nevertheless, our results are of the same nature as the estimate (4): While the dynamical quantity $h_{\text{top}}(\varphi)$ is replaced by the slow volume growths $s_1(\varphi)$ or $s_d(\varphi)$ and $l(\varphi)$, the homological quantity $\log \rho(\varphi_*)$ is replaced by *Floer-homological* quantities. In Theorems 1 and 2, this quantity is essentially the polynomial growth rate of the difference of the symplectic action of two closed orbits which represent generators of the symplectic Floer homology of φ_H , and in Theorem 3 this quantity is the polynomial growth rate of the rank of a Lagrangian Floer homology associated with φ . In the case of odd-dimensional spheres, the lower bound 1 in Theorem 3 can also be obtained by computing the homological growth of ϑ , and so in this case Theorem 3 is just a version of (4), see Subsection 5.7.

3. SLOW ENTROPY ON $\text{Symp}_0^c(M, d\lambda)$

We consider a symplectic manifold (M, ω) with or without boundary ∂M . Denote by $\text{Symp}_0^c(M, \omega)$ the identity component of the group $\text{Symp}^c(M, \omega)$ of symplectomorphisms of (M, ω) whose support is compact and contained in $M \setminus \partial M$, and denote by $\text{Ham}^c(M, \omega)$ its subgroup consisting of symplectomorphisms generated by Hamiltonian functions $H: S^1 \times M \rightarrow \mathbb{R}$ whose support is compact and contained in $S^1 \times (M \setminus \partial M)$. We shall only consider such Hamiltonians. We define the slow volume growth $s_1(\varphi)$ of $\varphi \in \text{Symp}^c(M, \omega)$ as in (2). A symplectic manifold (M, ω) is called *exact* if there exists a 1-form λ on M such that $\omega = d\lambda$. The boundary ∂M of a $2d$ -dimensional symplectic manifold (M, ω) is said to be *convex* if there exists a Liouville vector

field Y (i.e., $\mathcal{L}_Y\omega = d\iota_Y\omega = \omega$) which is defined near ∂M and is everywhere transverse to ∂M , pointing outwards. Equivalently, there exists a 1-form α on ∂M such that $d\alpha = \omega|_{\partial M}$ and such that $\alpha \wedge (d\alpha)^{d-1}$ is a volume form inducing the boundary orientation of $\partial M \subset M$. Following [13] we say that a symplectic manifold (M, ω) is *convex at infinity* if there exists an increasing sequence of compact submanifolds $M_i \subset M$ with smooth convex boundaries exhausting M , that is,

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots \subset M \quad \text{and} \quad \bigcup_i M_i = M.$$

Cotangent bundles T^*B over a closed base B are exhausted by the compact submanifolds T_i^*B with convex boundary, and so Theorem 1 follows from

Theorem 3.1. *Assume that $(M, d\lambda)$ is an exact symplectic manifold and that $\varphi \in \text{Symp}_0^c(M, d\lambda) \setminus \{\text{id}\}$. If φ is Hamiltonian, assume in addition that $(M, d\lambda)$ is convex at infinity. Then $s_1(\varphi) \geq 1$.*

3.1. Proof of Theorem 1. The main ingredient in the proof of Theorem 3.1 is the following result whose proof in [20] relies on symplectic Floer homology.

Proposition 3.2. *Assume that $(M, d\lambda)$ is an exact symplectic manifold convex at infinity. For any Hamiltonian function $H: S^1 \times M \rightarrow \mathbb{R}$ generating $\varphi \in \text{Ham}^c(M, d\lambda) \setminus \{\text{id}\}$ there exist $x, y \in \mathcal{P}_H$ such that $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.*

Here, \mathcal{P}_H denotes the set of 1-periodic orbits of φ_H^t , and the symplectic action $\mathcal{A}_H(x)$ of $x \in \mathcal{P}_H$ is defined as in (3). In the remainder of the proof we closely follow the proof of Theorem 1.4.A in [39]. Consider an exact symplectic manifold $(M, d\lambda)$. If ∂M is not empty, we replace M by its interior $M \setminus \partial M$, which we denote again by M . According to [33, Chapter 10], the map

$$(5) \quad \text{Flux}: \text{Symp}_0^c(M) \rightarrow H_c^1(M; \mathbb{R}), \quad \varphi \mapsto [\varphi^*\lambda - \lambda],$$

is a homomorphism which fits into the exact sequence

$$(6) \quad 0 \rightarrow \text{Ham}^c(M) \rightarrow \text{Symp}_0^c(M) \rightarrow H_c^1(M; \mathbb{R}) \rightarrow 0.$$

Case 1. φ is Hamiltonian. Assume that φ is generated by $H: S^1 \times M \rightarrow \mathbb{R}$. According to Proposition 3.2 we find $x, y \in \mathcal{P}_H$ such that

$$c := \mathcal{A}_H(y) - \mathcal{A}_H(x) > 0.$$

Choose a smoothly embedded curve $\sigma: [0, 1] \rightarrow M$ such that $\sigma(0) = x(0)$ and $\sigma(1) = y(0)$. For each $n \geq 1$ let l_n be the piecewise smooth loop $\varphi^n(\sigma) \cup -\sigma$.

Proposition 3.3. $\int_{l_n} \lambda = nc.$

Proof. We start with

Lemma 3.4. $\int_{l_n} \lambda = n \int_{l_1} \lambda.$

Proof. Since φ is Hamiltonian, so is φ^k for any $k \geq 1$, and so $\langle \text{Flux } \varphi^k, l_1 \rangle = 0$ for any $k \geq 1$. Therefore,

$$\int_{\varphi^{k+1}\sigma} \lambda - \int_{\varphi^k\sigma} \lambda = \int_{\varphi^k l_1} \lambda = \int_{l_1} (\varphi^k)^* \lambda = \int_{l_1} \lambda = \int_{\varphi\sigma} \lambda - \int_{\sigma} \lambda$$

and so

$$\begin{aligned} \int_{l_n} \lambda &= \int_{\varphi^n \sigma} \lambda - \int_{\sigma} \lambda = \sum_{k=0}^{n-1} \left(\int_{\varphi^{k+1}\sigma} \lambda - \int_{\varphi^k\sigma} \lambda \right) \\ &= \sum_{k=0}^{n-1} \left(\int_{\varphi\sigma} \lambda - \int_{\sigma} \lambda \right) = n \int_{l_1} \lambda. \end{aligned}$$

□

Lemma 3.5. $\int_{l_1} \lambda = \mathcal{A}_H(y) - \mathcal{A}_H(x).$

Proof. Define a 2-chain $\Delta: [0, 1] \times [0, 1] \rightarrow M$ by $\Delta(t, s) = \varphi_H^t \sigma(s)$. Assuming the boundary of $[0, 1] \times [0, 1]$ to be oriented counterclockwise we have

$$\partial\Delta = -\sigma - y + \varphi\sigma + x.$$

A computation given in the proof of Proposition 2.4.A in [39] shows that

$$\int_{\Delta} \omega = \int_0^1 H(t, x(t)) dt - \int_0^1 H(t, y(t)) dt.$$

Putting everything together we conclude that

$$\begin{aligned} \int_{l_1} \lambda &= \int_{\varphi\sigma} \lambda - \int_{\sigma} \lambda = \int_{\partial\Delta} \lambda - \int_x \lambda + \int_y \lambda \\ &= \int_{\Delta} \omega - \int_x \lambda + \int_y \lambda = \mathcal{A}_H(y) - \mathcal{A}_H(x), \end{aligned}$$

as claimed. □

We finally conclude from Lemmata 3.4 and 3.5 that

$$\int_{l_n} \lambda = n \int_{l_1} \lambda = nc,$$

and so the proof of Proposition 3.3 is complete. □

We measure the lengths of curves in M with respect to a Riemannian metric on M . Let $x, y \in \mathcal{P}_H$ and $l_n = \varphi^n \sigma \cup -\sigma$ be as above. Choose $C < \infty$ so large that $|\lambda(x)| \leq C$ for all $x \in \text{supp } \varphi \cup \sigma$. Then $l_n \subset \text{supp } \varphi \cup \sigma$ for all $n \geq 1$ and so

$$nc = \int_{l_n} \lambda \leq C \text{length } l_n,$$

whence $\frac{c}{C}n \leq \text{length } \varphi^n \sigma + \text{length } \sigma$ for all $n \geq 1$. We conclude that

$$\begin{aligned} s_1(\varphi) &\geq \liminf_{n \rightarrow \infty} \frac{1}{\log n} \log \text{length } \varphi^n \sigma \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{\log n} \log \left(\frac{c}{C} n \right) \\ &= 1. \end{aligned}$$

Case 2: φ is not Hamiltonian. In this case, $\text{Flux } \varphi \in H_c^1(M; \mathbb{R})$ does not vanish. Let $H_1^{cl}(M; \mathbb{R})$ be the first homology with closed support of M . Since the pairing

$$H_c^1(M; \mathbb{R}) \otimes H_1^{cl}(M; \mathbb{R}) \rightarrow \mathbb{R}, \quad ([\alpha], [\gamma]) \mapsto \int_{\gamma} \alpha,$$

is non-degenerate, we therefore find $[\gamma] \in H_1^{cl}(M; \mathbb{R})$ such that

$$\langle \text{Flux } \varphi, [\gamma] \rangle = \int_{\gamma} \varphi^* \lambda - \lambda =: c > 0.$$

Since $\varphi^* \lambda - \lambda$ has compact support, we can represent $[\gamma]$ by a smoothly embedded line $\gamma: \mathbb{R} \rightarrow M$ such that $\gamma \cap \text{supp } \varphi \subset \gamma([0, 1])$. Define $\sigma: [0, 1] \rightarrow M$ by $\sigma(t) = \gamma(t)$ and set again $l_n = \varphi^n \sigma \cup -\sigma$. Using that Flux is a homomorphism we then find

$$\begin{aligned} \int_{l_n} \lambda &= \int_{\sigma} (\varphi^n)^* \lambda - \lambda = \int_{\gamma} (\varphi^n)^* \lambda - \lambda \\ &= \langle \text{Flux } \varphi^n, [\gamma] \rangle = n \langle \text{Flux } \varphi, [\gamma] \rangle = nc. \end{aligned}$$

Proceeding as in Case 1 we conclude that $s_1(\varphi) \geq 1$. The proof of Theorem 3.1 is complete. \square

3.2. Distortion in finitely generated subgroups of $\text{Symp}_0^c(M, d\lambda)$.

We consider an exact symplectic manifold $(M, d\lambda)$ which is convex at infinity. Theorem 3.1 yields at once

Corollary 3.6. *The group $\text{Symp}_0^c(M, d\lambda)$ has no torsion.*

Proof. Assume that $\varphi^m = \text{id}$ for some $\varphi \in \text{Symp}_0^c(M, d\lambda)$ and $m \geq 1$. Using definition (2) of $s_1(\varphi)$ and $\varphi^{mn}(\sigma) = \sigma$ for all $n \geq 1$ and $\sigma \in \Sigma_1$ we then find

$$\begin{aligned} s_1(\varphi) &= \sup_{\sigma \in \Sigma_1} \liminf_{n \rightarrow \infty} \frac{\log \mu_g(\varphi^n(\sigma))}{\log n} \\ &\leq \sup_{\sigma \in \Sigma_1} \liminf_{n \rightarrow \infty} \frac{\log \mu_g(\varphi^{mn}(\sigma))}{\log mn} \\ &= 0, \end{aligned}$$

and so Theorem 3.1 implies $\varphi = \text{id}$. \square

Proposition 3.2 can be used to obtain deeper insight into the algebraic structure of the groups $\text{Ham}^c(M, d\lambda)$ and $\text{Symp}_0^c(M, d\lambda)$. Following [39] we consider a finitely generated subgroup \mathcal{G} of $\text{Symp}_0^c(M, d\lambda)$. Fix a set of generators of \mathcal{G} and denote by $\|\varphi\|$ the word length of $\varphi \in \mathcal{G}$. The *distortion* $d(\varphi) \in [0, 1]$ of $\varphi \in \mathcal{G}$ defined as

$$d(\varphi) = 1 - \liminf_{n \rightarrow \infty} \frac{\log \|\varphi^n\|}{\log n}$$

does not depend on the set of generators.

Theorem 3.7. *Consider a finitely generated subgroup \mathcal{G} of $\text{Symp}_0^c(M, d\lambda)$.*

- (i) *If $\mathcal{G} \subset \text{Ham}^c(M, d\lambda)$, then $d(\varphi) = 0$ for all $\varphi \in \mathcal{G} \setminus \{\text{id}\}$.*
- (ii) *If $\mathcal{G} \subset \text{Symp}_0^c(M, d\lambda)$, then $d(\varphi) \leq \frac{1}{2}$ for all $\varphi \in \mathcal{G} \setminus \{\text{id}\}$, and $d(\varphi) = 0$ if φ is not Hamiltonian.*

Question 3.8. Can the estimate $d(\varphi) \leq \frac{1}{2}$ in Theorem 3.7 (ii) be replaced by $d(\varphi) = 0$?

Sketch of the proof of Theorem 3.7: The proof can be extracted from [39], where an analogous result was found for closed symplectic manifolds (M, ω) with $\pi_2(M) = 0$. In our situation the arguments are considerably easier, however. Choosing k so large that $\mathcal{G} \subset \text{Symp}_0^c(M_k, d\lambda)$ we can assume that $(M, d\lambda) = (M_k, d\lambda)$. In addition to the geometric arguments in [39, Sections 4.1–4.5] one uses that the action $\mathcal{A}_H(x)$ of a contractible $x \in \mathcal{P}_H$ depends only on $x(0)$ and $\varphi_H \in \text{Ham}^c(M_k, d\lambda)$ (see [7, Remark 3.1.1] or [20, Corollary 6.2]), that λ is bounded with respect to any Riemannian metric on M_k , as well as Proposition 3.2, Lemmata 3.4 and 3.5, and the flux homomorphism (5). \square

Second proof of Corollary 3.6: Let \mathcal{G} be the cyclic subgroup generated by $\varphi \in \text{Symp}_0^c(M, d\lambda)$. If $\varphi^m = \text{id}$ for some $m \neq 0$, then $d(\varphi) = 1$, and

so Theorem 3.7 yields $\varphi = \text{id}$. \square

Following again [39] we notice that Theorem 3.7 (ii) can also be used to obtain restrictions for representations of discrete groups on $\text{Symp}_0^c(M, d\lambda)$. An element x of an abstract finitely generated group \mathcal{G} is called a **U-element** if it is of infinite order and

$$\liminf_{n \rightarrow \infty} \frac{\log \|x^n\|}{\log n} = 0.$$

Theorem 3.7 (ii) shows that $\phi(x) = \text{id}$ for every homomorphism $\phi: \mathcal{G} \rightarrow \text{Symp}_0^c(M, d\lambda)$. An example of a **U-element** is the element a of the Baumslag–Solitar group

$$BS(q, p) = \langle a, b \mid a^q = ba^pb^{-1} \rangle, \quad q, p \in \mathbb{Z} \setminus \{0\}, \quad |p| < |q|,$$

see [39, Example 1.6.E]. Other examples of finitely generated groups containing a **U-element** are $SL(n; \mathbb{Z})$ for $n \geq 3$ and, more generally, irreducible non-uniform lattices in a semisimple real Lie group whose real rank is at least two and which is connected, without compact factors and with finite centre. Let \mathcal{G} be such a lattice. As in [39, 1.6.J] one obtains

Corollary 3.9. *Every homomorphism $\mathcal{G} \rightarrow \text{Symp}_0^c(M, d\lambda)$ has finite image.*

Remarks 3.10. **1.** Combining the proof of Theorem 9.1.6 in [34] with the compactness theorems for J -holomorphic curves in geometrically bounded symplectic manifolds [25, 56] one sees that Proposition 3.2 and hence Theorem 3.1 and the results in Subsection 3.2 extend to geometrically bounded exact symplectic manifolds.

2. Consider the full group $\text{Symp}^c(T^*S^d, d\lambda)$ of compactly supported symplectomorphisms of the standard cotangent bundle $(T^*S^d, d\lambda)$, where $d = 1, 2$. Recall that $\text{Symp}^c(T^*S^d, d\lambda) / \text{Symp}_0^c(T^*S^d, d\lambda)$ is infinite cyclic, [45]. Corollary 3.6 thus implies that $\text{Symp}^c(T^*S^d, d\lambda)$ has no torsion. Moreover, given a lattice \mathcal{G} as in Corollary 3.9, every homomorphism $\mathcal{G} \rightarrow \mathbb{Z}$ is trivial because \mathcal{G} has property (T), see [12]; together with Corollary 3.9 we thus see that every homomorphism $\mathcal{G} \rightarrow \text{Symp}^c(T^*S^d, d\lambda)$ has finite image.

3.3. Proof of Proposition 1. We first consider the open ball $B^{2n}(1)$ of radius 1 in \mathbb{R}^{2n} . Choose a smooth function $f: \mathbb{R} \rightarrow [0, 1]$ such that

$$f(r) = \begin{cases} 1 & \text{iff } r \leq \frac{1}{4n}, \\ 0 & \text{iff } r \geq 1 - \frac{1}{4n}, \end{cases}$$

and set $H(x) = f(|x|^2)$. Denoting by $J \in \mathcal{L}(\mathbb{R}^{2n})$ the standard complex structure on \mathbb{R}^{2n} we find that the time- n -map of the Hamiltonian flow of H is given by

$$\varphi^n(x) = e^{2f'(|x|^2)nJ}x, \quad x \in B^{2n}(1).$$

Notice that φ^n preserves the Euclidean length of curves contained in a round sphere. Moreover, there exists a constant $C < \infty$ such that

$$\|d\varphi^n(x)\| \leq Cn \quad \text{for all } x \in B^{2n}(1) \text{ and } n \geq 1,$$

and so $s_i(\varphi) \leq 1$ for all i .

In order to show that $s_i(\varphi) \geq 1$ for $i \in \{1, \dots, 2n-1\}$ we shall investigate the φ -orbits of particular cubes σ_i , which for convenience are defined on $Q'_i := [0, \frac{1}{i+1}]^i$ instead on $[0, 1]^i$. We measure the size of the cubes $\varphi^n(\sigma_i)$ with respect to the Euclidean metric, and we use coordinates $(x_1, y_1, \dots, x_n, y_n)$ in \mathbb{R}^{2n} . We abbreviate

$$t = (t_1, \dots, t_i) \quad \text{and} \quad \mathbf{t} = \sqrt{t_1^2 + \dots + t_i^2}.$$

If $i \in \{1, \dots, 2n-1\}$ is odd, we define $\sigma_i: Q'_i \hookrightarrow B^{2n}(1)$ by

$$\sigma_i(t) = \begin{cases} (t, 0, \dots, 0) & \text{if } i = 1, \\ (t_1, 0, t_2, t_3, \dots, t_{i-1}, t_i, 0, \dots, 0) & \text{if } i \geq 3. \end{cases}$$

Then $\varphi^n(\sigma_i(t)) = e^{2f'(\mathbf{t}^2)nJ}\sigma_i(t)$. A computation shows that

$$\mu(\varphi^n(\sigma_i)) = \int_{Q'_i} \sqrt{1 + (4nf''(\mathbf{t}^2)t_1^2\mathbf{t}^2)^2} dt.$$

By our choice of f , this expression grows like n , and so $s_i(\varphi) \geq 1$.

If $i \in \{2, \dots, 2n-2\}$ is even, we define $\sigma_i: Q'_i \hookrightarrow B^{2n}(1)$ by

$$\sigma_i(t) = \begin{cases} (t_1, 0, t_2, 0, \dots, 0) & \text{if } i = 2, \\ (t_1, 0, t_2, 0, t_3, t_4, \dots, t_{i-1}, t_i, 0, \dots, 0) & \text{if } i \geq 4. \end{cases}$$

A computation shows that

$$\mu(\varphi^n(\sigma_i)) = \int_{Q'_i} \sqrt{1 + (4nf''(\mathbf{t}^2)(t_1^2 + t_2^2)\mathbf{t}^2)^2} dt.$$

By our choice of f , this expression grows like n , and so $s_i(\varphi) \geq 1$.

Assume now that (M, ω) is an arbitrary symplectic manifold. By Darboux's theorem, there exists $\epsilon > 0$ and a symplectic embedding χ of the open ball $B^{2n}(\epsilon)$ of radius ϵ into M , and proceeding as above we find a compactly supported Hamiltonian diffeomorphism φ of $B^{2n}(\epsilon)$

such that $s_i(\varphi) = 1$ for all $i \in \{1, \dots, 2n-1\}$. Define the Hamiltonian diffeomorphism ψ of M by

$$\psi(x) = \begin{cases} \chi\varphi\chi^{-1}(x) & \text{if } x \in \chi(B^{2n}(\epsilon)), \\ x & \text{if } x \notin \chi(B^{2n}(\epsilon)). \end{cases}$$

Then $s_i(\psi) = s_i(\varphi) = 1$ for all $i \in \{1, \dots, 2n-1\}$, and so the proof of Proposition 1 is complete. \square

4. PROOF OF THEOREM 2

4.1. Unboundedness of the action functional. Consider a closed manifold B whose fundamental group $\pi_1(B)$ is finite or has infinitely many conjugacy classes, and let $H: S^1 \times T^*B \rightarrow \mathbb{R}$,

$$H(t, q, p) = \frac{1}{2} |p - A(t, q)|^2 + V(t, q),$$

be a classical Hamiltonian function. Here, $|\cdot|$ refers again to the Riemannian metric on T^*B induced by a fixed Riemannian metric on B . The main ingredient of the proof of Theorem 2 is

Proposition 4.1. *The action functional \mathcal{A}_H is not bounded from above on \mathcal{P}_H .*

Proof. Since H is fibrewise convex, its Legendre transform $L: S^1 \times TB \rightarrow \mathbb{R}$ is defined and equals

$$(7) \quad L(t, q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 + \langle B(t, q), \dot{q} \rangle + W(t, q)$$

where $B(t, q) = -A(t, q)$ and $W(t, q) = \frac{1}{2} |A(t, q)|^2 - V(t, q)$. The 1-periodic orbits $x(t)$ of the flow of H correspond to 1-periodic orbits $q(t)$ of the Lagrangian flow generated by L , and $\mathcal{A}_H(x)$ equals the Lagrangian action

$$\mathcal{L}_H(q) = \int_0^1 L(t, q, \dot{q}) dt.$$

It is proved in [5]¹ that if $\pi_1(B)$ is finite, then \mathcal{L}_H is not bounded from above on the set of 1-periodic orbits, and so the same holds for \mathcal{A}_H . Assume now that $\pi_1(B)$ has infinitely many conjugacy classes. These conjugacy classes correspond to the connected components of the space ΩB of continuous maps $S^1 \rightarrow B$, and ΩB is homotopy equivalent to the

¹Lemma 2.6 (c) in [5] should, however, read

$$\int_0^1 \mathcal{E}(\beta(\lambda))^{1/2} d\lambda \leq d_\beta + \mathcal{E}(\beta(0))^{1/2},$$

and then the proofs of Lemmata 2.7 and 4.3 should be corrected accordingly.

Sobolev space $\Omega^1 B = W^1(S^1, B)$ of maps $S^1 \rightarrow B$ with “square integrable derivative”, see [5]. It is proved in [5] that \mathcal{L}_H is a C^1 -functional on $\Omega^1 B$ and that each of the infinitely many components of $\Omega^1 B$ contains a critical point q_n of \mathcal{L}_H , $n = 1, 2, \dots$. Since these loops are continuous, they are smooth 1-periodic orbits of the Lagrangian flow. Arguing by contradiction we assume that $\{\mathcal{L}_H(q_n)\} \subset \mathbb{R}$ is bounded from above. According to Lemma 2.4 in [5], \mathcal{L}_H is bounded from below on $\Omega^1 B$, and so $\{\mathcal{L}_H(q_n)\}$ is bounded; moreover, $\nabla \mathcal{L}_H(q_n) = 0$ for all n . Since \mathcal{L}_H satisfies the Palais–Smale condition on $\Omega^1 B$, [5], we conclude that the sequence (q_n) has a convergent subsequence in $\Omega^1 B$, which is impossible. This contradiction completes the proof of Proposition 4.1. \square

4.2. End of the proof. Let B and H be as in Theorem 2. In view of Proposition 4.1 we find $x, y \in \mathcal{P}_H$ such that $c := \mathcal{A}_H(y) - \mathcal{A}_H(x) > 0$. Choose a smoothly embedded curve $\sigma: [0, 1] \rightarrow T^*B$ such that $\sigma(0) = x(0)$ and $\sigma(1) = y(0)$. For each $n \geq 1$ let l_n be the piecewise smooth loop $\varphi_H^n(\sigma) \cup -\sigma$. According to Proposition 3.3 we then have

$$(8) \quad \int_{l_n} \lambda = nc.$$

We measure the lengths of curves in T^*B with respect to g^* .

Proof of Theorem 2 (i). We choose $R > 0$ so large that

$$(9) \quad x(0), y(0) \in T_R^*B.$$

Claim 4.2. $\text{length } l_n \geq \min(c, 1)\sqrt{n}$ for any $n \geq 4R^2$.

Proof. Fix $n \geq 4R^2$. We distinguish two cases.

Case 1: $l_n \subset T_{\sqrt{n}}^*B$.

We have $|\lambda(l_n(t))| \leq \sqrt{n}$ for all $t \in [0, 1]$. Together with the identity (8) we infer that

$$nc = \int_{l_n} \lambda \leq \sqrt{n} \text{length } l_n,$$

and so $\text{length } l_n \geq c\sqrt{n}$.

Case 2: $l_n(t) \notin T_{\sqrt{n}}^*B$ for some $t \in [0, 1]$.

In view of (9) and the assumption $n \geq 4R^2$ we can estimate

$$\begin{aligned} \text{length } l_n &\geq (|p(l_n(t))| - |p(x(0))|) + (|p(l_n(t))| - |p(y(0))|) \\ &\geq (\sqrt{n} - R) + (\sqrt{n} - R) \\ &\geq \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2}, \end{aligned}$$

and so $\text{length } l_n \geq \sqrt{n}$. \square

According to Claim 4.2,

$$\min(c, 1)\sqrt{n} \leq \text{length } \sigma + \text{length } \varphi_H^n \sigma$$

for all $n \geq 4R^2$, and so

$$\begin{aligned} s_1(\varphi) &\geq \liminf_{n \rightarrow \infty} \frac{1}{\log n} \log \text{length } \varphi_H^n \sigma \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{\log n} \log (\min(c, 1)\sqrt{n}) \\ &= \frac{1}{2}. \end{aligned}$$

Proof of Theorem 2 (ii). Let $r < \infty$ be such that H is independent of t on $T^*B \setminus T_r^*B$. Choosing r larger if necessary we can assume that $\sigma \subset T_r^*B$. Choose a so large that T_r^*B is contained in the sublevel set

$$H^a = \{(t, q, p) \in S^1 \times T^*B \mid H(t, q, p) \leq a\}.$$

Since H is time-independent on the boundary of H^a , the set H^a is invariant under φ_H , and so $\varphi_H^n \sigma \subset H^a$ for all n . Choosing $R < \infty$ so large that $H^a \subset T_R^*B$, we then have $l_n \subset T_R^*B$ for all $n \geq 1$. In particular, $|\lambda(l_n(t))| \leq R$ for all $n \geq 1$ and all $t \in [0, 1]$. Together with (8) we infer that $\text{length } l_n \geq \frac{c}{R}n$ for all $n \geq 1$. Proceeding as in the proof of (i) we conclude that $s_1(\varphi_H) \geq 1$. The proof of Theorem 2 is complete. \square

Remarks 4.3. (i) The results from [5] used in the proof of Proposition 4.1 hold for more general convex Lagrangians than classical ones of the form (7), and so Proposition 4.1 and hence Theorem 2 hold for more general convex Hamiltonians.

(ii) Using [8] and [9, Section 6] supplemented by a recent result in [1], one finds that Proposition 4.1 and hence Theorem 2 hold for a yet larger class of Hamiltonians H , which do not need to be convex. They are only assumed to satisfy the two asymptotic conditions

$$(H1) \quad dH(X)(t, q, p) - H(t, q, p) \geq c|p|^2 - C,$$

$$(H2) \quad |\nabla_q H(t, q, p)| \leq C(1 + |p|^2) \quad \text{and} \quad |\nabla_p H(t, q, p)| \leq C(1 + |p|),$$

for all $(t, q, p) \in S^1 \times T^*B$. Here, $X = \sum_i p_i \frac{\partial}{\partial p_i}$ is the Liouville vector field, ∇ denotes the Levi-Civita connection with respect to the Riemannian metric g on B , and c and C are positive constants depending only on g .

5. PROOF OF THEOREM 3

In this section we shall first describe the known Riemannian manifolds with periodic geodesic flow and then define twists on such manifolds. We then prove Theorem 3 and Corollary 3, and finally study twists from a topological point of view.

5.1. P -manifolds. Geodesics of a Riemannian manifold will always be parametrized by arc-length. A P -manifold is by definition a connected Riemannian manifold all whose geodesics are periodic. It follows from Wadsley's Theorem that the geodesics of a P -manifold admit a common period, see [60] and [6, Lemma 7.11]. We normalize the Riemannian metric such that the minimal common period is 1. Every P -manifold is closed, and besides S^1 every P -manifold has finite fundamental group, see [6, 7.37]. The main examples of P -manifolds are the CROSSes

$$S^d, \quad \mathbb{RP}^d, \quad \mathbb{CP}^n, \quad \mathbb{HP}^n, \quad \mathbb{CaP}^2$$

with their canonical Riemannian metrics suitably normalized. The simplest way of obtaining other P -manifolds is to look at Riemannian quotients of CROSSes. The main examples thus obtained are the spherical space forms S^{2n+1}/G where G is a finite subgroup of $O(2n+2)$ acting freely on S^{2n+1} . These spaces are classified in [59], and examples are lens spaces, which correspond to cyclic G . According to [2, pp. 11–12] and [6, 7.17 (c)], the only other Riemannian quotients of CROSSes are the spaces $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$; here, the fixed point free involution on \mathbb{CP}^{2n-1} is induced by the involution

$$(10) \quad (z_1, z'_1, \dots, z_n, z'_n) \mapsto (\bar{z}'_1, -\bar{z}_1, \dots, \bar{z}'_n, -\bar{z}_n)$$

of \mathbb{C}^{2n} . Notice that $\mathbb{CP}^1/\mathbb{Z}_2 = \mathbb{RP}^2$. We shall thus assume $n \geq 2$. On spheres, there exist P -metrics which are not isometric to the round metric g_{can} . We say that a P -metric on S^d is a *Zoll metric* if it can be joined with g_{can} by a smooth path of P -metrics. All known P -metrics on S^d are Zoll metrics. For each $d \geq 2$, the Zoll metrics on S^d form an infinite dimensional space. For $d \geq 3$, the known Zoll metrics admit $SO(d)$ as isometry group, but for $d = 2$, the set of Zoll metrics contains an open set all of whose elements have trivial isometry group. We refer to [6, Chapter 4] for more information about Zoll metrics.

CROSSes, their quotients and Zoll manifolds are the only known P -manifolds. It would be interesting to know whether this list is complete. As an aside, we mention that for the known P -manifolds all geodesics are simply closed. Whether this is so for all P -manifolds is unknown, [6, 7.73 (f''')], except for P -metrics on the 2-sphere, [24].

For a geodesic $\gamma: \mathbb{R} \rightarrow B$ of a P -manifold (B, g) and $t > 0$ we let $\text{ind } \gamma(t)$ be the number of linearly independent Jacobi fields along $\gamma(s)$, $s \in [0, t]$, which vanish at $\gamma(0)$ and $\gamma(t)$. If $\text{ind } \gamma(t) > 0$, then $\gamma(t)$ is said to be conjugate to $\gamma(0)$ along γ . The index of γ defined as

$$\text{ind } \gamma = \sum_{t \in]0, 1[} \text{ind } \gamma(t)$$

is a finite number. According to [6, 1.98 and 7.25], every geodesic on (B, g) has the same index, say k . We then call (B, g) a P_k -manifold.

Proposition 5.1. *For the known P -manifolds, the indices of geodesics are as follows.*

(B, g)	S^d	\mathbb{RP}^d	\mathbb{CP}^n	\mathbb{HP}^n	\mathbb{CaP}^2
k	$d - 1$	0	1	3	7

For a quotient S^{2n+1}/G we have $k = 0$ if $-\text{id} \in G$ and $k = 2n$ otherwise, and for $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$ we have $k = 1$. Finally, for a Zoll manifold modelled on S^d we have $k = d - 1$.

Proof. For CROSSes, the result is well known, see [6, 3.35 and 3.70]. For this proof, we assume the Riemannian metrics on S^{2n+1}/G and $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$ to be locally isometric to the normalized Riemannian metrics on S^{2n+1} and \mathbb{CP}^{2n-1} , respectively. The Jacobi fields on S^{2n+1}/G and $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$ then correspond to Jacobi fields on S^{2n+1} and \mathbb{CP}^{2n-1} . It follows that if all geodesics on S^{2n+1}/G of length $1/2$ are closed, i.e., $-\text{id} \in G$, then $k = 0$, and $k = 2n$ otherwise. The isometry σ of \mathbb{CP}^{2n-1} induced by (10) maps a point x to its conjugate locus diffeomorphic to \mathbb{CP}^{2n-2} , and the family of geodesics emanating from x which pass through $\sigma(x)$ is 2-dimensional. Since $n \geq 2$, there are geodesics not passing through $\sigma(x)$, and so $k = 1$. Finally, let g be a Zoll metric on S^d . In order to show $k = d - 1$, we choose a smooth family g_t of P -metrics on S^d such that $g_0 = g_{\text{can}}$ and $g_1 = g$. Fix $x \in S^d$ and $v \in T_x S^d$ with $|v|_0 = 1$. For $t \in [0, 1]$ let γ_t be the geodesic of (S^d, g_t) with $\gamma_t(0) = x$ and $\dot{\gamma}_t(0) = \frac{|v|_0}{|v|_t} v$. The set $T = \{t \in [0, 1] \mid \text{ind } \gamma_t = d - 1\}$ contains 0, and comparing Jacobi fields on (S^d, g_t) along γ_t one easily shows that T is both open and closed. In particular, $\text{ind } \gamma_1 = d - 1$, and so (S^d, g_1) is a P_{d-1} -metric. \square

We conclude this subsection by proving a property shared by all P -manifolds (B, g) . For each point v in the unit tangent bundle $\partial T_1 B$

over B we denote by $l(v)$ the length of the simply closed orbit of the geodesic flow through v . By our normalization of g , $l(v) = 1/j$ for some integer j depending on v .

Lemma 5.2. *The set of point $v \in \partial T_1 B$ with $l(v) = 1$ is open and dense in $\partial T_1 B$.*

Proof. According to [6, Corollary A.18], the length function l is continuous on an open and dense subset V of $\partial T_1 B$. It follows from Proposition 4.5 of [59] that $l|_V \equiv 1/j$ for some integer j and that for each $v \in \partial T_1 B$ there is an integer h such that $l(v) = 1/(hj)$. Our choice of g implies that $j = 1$, and so $l(v) = 1$ for all $v \in V$. \square

5.2. Twists. Consider a P -manifold (B, g) . As before, we choose coordinates (q, p) on T^*B , and using g we identify the cotangent bundle T^*B with the tangent bundle TB . The Hamiltonian flow of the function $\frac{1}{2}|p|^2$ corresponds to the geodesic flow on TB . For any smooth function $f: [0, \infty[\rightarrow [0, \infty[$ such that

$$(11) \quad f(r) = 0 \text{ for } r \text{ near } 0 \quad \text{and} \quad f'(r) = 1 \text{ for } r \geq 1$$

we define the *twist* ϑ_f as the time-1-map of the Hamiltonian flow generated by $f(|p|)$. Since (B, g) is a P -manifold, ϑ_f is the identity on $T^*B \setminus T_1^*B$, and so $\vartheta_f \in \text{Symp}^c(T^*B)$.

Proposition 5.3. (i) *The class $[\vartheta_f] \in \pi_0(\text{Symp}^c(T^*B))$ does not depend on the choice of f .*

(ii) *$s_i(\vartheta_f^m) = l(\vartheta_f^m) = 1$ for every $i \in \{1, \dots, 2d-1\}$, every $m \in \mathbb{Z} \setminus \{0\}$ and every f .*

Proof. (i) Let $f_i: [0, 1] \rightarrow [0, 1]$, $i = 1, 2$, be two functions as in (11). Then the functions $f_s = (1-s)f_1 + sf_2$, $s \in [0, 1]$, are also of this form, and $s \mapsto \vartheta_{f_s}$ is an isotopy in $\text{Symp}^c(T^*B)$ joining ϑ_{f_1} with ϑ_{f_2} .

(ii) The proof is similar to the proof of Proposition 1. Without loss of generality we assume $m = 1$. Let ϑ^t be the Hamiltonian flow of $f(|p|)$. Then $\vartheta_f = \vartheta^1$. For each $r > 0$ the hypersurface $S_r = \partial T_r^*B$ is invariant under ϑ^t . We denote by ϑ_r^t the restriction of ϑ^t to S_r . As before, we endow T^*B with the Riemannian metric g^* . For $x \in S_r$ let $\|d\vartheta_r^t(x)\|$ be the operator norm of the differential of ϑ_r^t at x induced by g^* . Since ϑ_1^t is 1-periodic, we find $C < \infty$ such that $\|d\vartheta_1^t(x)\| \leq C$ for all t and all $x \in S_1$. Since

$$\vartheta_r^t(x) = r\vartheta_1^{f'(r)t}\left(\frac{x}{r}\right)$$

for all $t \in \mathbb{R}$, $r > 0$ and $x \in S_r$, we conclude that

$$(12) \quad \|d\vartheta_r^t(x)\| = \left\| d\vartheta_1^{f'(r)t} \left(\frac{x}{r} \right) \right\| \leq C$$

for all $t \in \mathbb{R}$, $r > 0$ and $x \in S_r$. We next fix $(q, p) \in T^*B \setminus B$ and consider the line $F_p = \mathbb{R}p \subset T_q^*B$ orthogonal to $S_{|p|}$ through (q, p) . We denote by ϑ_p^t the restriction of ϑ^t to F_p . Let γ be the geodesic on B with $\gamma(0) = q$ and $\dot{\gamma}(0) = \frac{p}{|p|}$. Then $\vartheta_p^t(F_p) \subset T^*\gamma$ for all t . As a parametrized curve, γ is isometric to the circle S^1 of length 1, and so $T^*\gamma \setminus \gamma$ is isometric to $T^*S^1 \setminus S^1$. We thus find

$$(13) \quad \|d\vartheta_p^t(q, p)\| = \sqrt{(f''(|p|)t)^2 + 1} \leq C'(t + 1)$$

for all $t \geq 0$ and $(q, p) \in T^*B$ and some constant $C' < \infty$. The estimates (12) and (13) show that for any i -cube $\sigma: Q^i \hookrightarrow T^*B$,

$$\mu_{g^*}(\vartheta_f^n(\sigma)) \leq C^{i-1}C'(n + 1)$$

for all $n \geq 1$, and so $s_i(\vartheta_f) \leq 1$ for all i .

We are left with showing $s_i(\vartheta_f) \geq 1$ for $i \in \{1, \dots, 2d - 1\}$. Fix $q \in B$, choose an orthonormal basis $\{e_1, \dots, e_d\}$ of T_q^*B , and let R be such that $f([0, 1]) \subset [0, R]$. For $j \in \{1, \dots, d\}$ we let E^j be the subspace of T_q^*B generated by $\{e_1, \dots, e_j\}$, and we set $E_R^j = E^j \cap T_R^*B$. We first assume $i \in \{1, \dots, d\}$. We then choose $\sigma: Q^i \hookrightarrow E^i$ such that $E_R^i \subset \sigma(Q^i)$. The set $\pi(\vartheta_f(\sigma))$ consists of a smooth $(i-1)$ -dimensional family of geodesics in B . Its i -dimensional measure $\mu_g(\pi(\vartheta_f(\sigma)))$ with respect to g thus exists and is positive. Moreover, $\pi(\vartheta_f^n(\sigma)) = \pi(\vartheta_f(\sigma))$ for all $n \geq 1$, and every point in $\pi(\vartheta_f^n(\sigma))$ has at least $2n$ preimages in $\vartheta_f^n(\sigma)$. Since $\pi: (T^*B, g^*) \rightarrow (B, g)$ is a Riemannian submersion, we conclude that

$$\mu_{g^*}(\vartheta_f^n(\sigma)) \geq 2n\mu_g(\pi(\vartheta_f^n(\sigma))),$$

and so $s_i(\vartheta_f) \geq 1$.

Since the fibre T_q^*B is a Lagrangian submanifold of T^*B , we have shown that $s(\vartheta_f) = l(\vartheta_f) = 1$. We shall therefore only sketch the proof of the remaining inequalities $s_i(\vartheta_f) \geq 1$, $i \in \{d+1, \dots, 2d-1\}$. For such an i we choose a small $\epsilon > 0$, set $B_i = \exp_q E_\epsilon^{i-d}$, and choose $\sigma: Q^i \hookrightarrow T^*B_i$ such that $T_R^*B_i \subset \sigma(Q^i)$. Then there is a constant $c > 0$ such that

$$(14) \quad \mu_{g^*}(\vartheta_f^n(\sigma)) \geq cn \quad \text{for all } n \geq 1.$$

This is so because ϑ_f restricts to a symplectomorphism on the $i-d$ cylinders $T^*\gamma_j$ over the geodesics γ_j with $\gamma_j(0) = q$ and $\dot{\gamma}(0) = e_j$, and – as we have seen above – grows linearly on the $(2d-i)$ -dimensional

remaining factor in the fibre. An explicit proof of (14) can be given by computing the differential $d\vartheta_f(q, p)$ with respect to suitable orthonormal bases of $T_{(q,p)}T^*B$ and $T_{\vartheta_f(q,p)}T^*B$. \square

Let $\tau \in \text{Symp}^c(T^*S^d)$ be a generalized Dehn twist as defined in Figure 2; for an analytic definition we refer to [47, 5a]. Then τ^2 is a twist ϑ_f . Proposition 5.3 (ii) and the argument given in 5.6 below thus show that $s_i(\tau^m) = l(\tau^m) = 1$ for every $i \in \{1, \dots, 2d-1\}$ and every $m \in \mathbb{Z} \setminus \{0\}$.

Theorem 3 is a special case of the following theorem, which is the main result of this section.

Theorem 5.4. *Let (B, g) be a d -dimensional P_k -manifold, and let ϑ be a twist on T^*B . If $d = 2$ and B is diffeomorphic to \mathbb{RP}^2 , assume that $g = g_{\text{can}}$, and if $d \geq 3$ and $k = 1$, assume that (B, g) is \mathbb{CP}^n or $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$. If $\varphi \in \text{Symp}^c(T^*B)$ is such that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(T^*B))$ for some $m \in \mathbb{Z} \setminus \{0\}$, then $s_d(\varphi) \geq l(\varphi) \geq 1$.*

Remarks 5.5. **1.** Theorem 5.4 covers all known P -manifolds. Indeed, the only known P -metric on \mathbb{RP}^2 is g_{can} , and the only known P_1 -manifolds of dimension at least 3 are \mathbb{CP}^n and $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$. The following two results suggest that Theorem 5.4 in fact covers all P -manifolds.

(i) If g is a P -metric on \mathbb{RP}^2 such that for some point x there exists $l > 0$ such that all geodesics of length l emanating from x are embedded circles, then $g = g_{\text{can}}$ by Green's theorem [23].

(ii) If (B, g) is a P_1 -manifold containing a point x for which there exists $l > 0$ such that for each geodesic γ emanating from x , $\gamma(l) = x$ and $\gamma(t) \neq x$ for all $t \in (0, l)$, then B has the homotopy type of \mathbb{CP}^n , see [6, 7.23].

2. (i) We recall from [45, 52] that $\pi_0(\text{Symp}^c(T^*S^2)) = \mathbb{Z}$ is generated by the class $[\tau]$ of a generalized Dehn twist.

(ii) For S^d , $d \geq 3$, $[\tau]^2 = [\vartheta]$ and Theorem 5.4 imply that $[\tau]$ generates an infinite cyclic subgroup of $\pi_0(\text{Symp}^c(T^*S^d))$, and for those P_k -manifold (B, g) covered by Theorem 5.4 which are not diffeomorphic to a sphere, Theorem 5.4 implies that $[\vartheta]$ generates an infinite cyclic subgroup of $\pi_0(\text{Symp}^c(T^*B))$. This was proved in [47, Corollary 4.5] for all P -manifolds.² It would be interesting to know whether there

²It is assumed in [47] that $H^1(B; \mathbb{R}) = 0$. Besides for $B = S^1$ this is, however, guaranteed by the Bott–Samelson Theorem [6, Theorem 7.37].

are other elements in these symplectic mapping class groups. \diamond

Theorem 5.4 is proved in the next two subsections.

5.3. Lagrangian Floer homology. Floer homology for Lagrangian submanifolds was invented by Floer in a series of seminal papers, [15, 16, 17, 18], and more general versions have been developed meanwhile, [36, 21]. In this subsection we first follow [31] and define Lagrangian Floer homology for certain pairs of Lagrangian submanifolds with boundary in an exact compact convex symplectic manifold. We then compute this Floer homology in the special case that the pair consists of a fibre and the image of another fibre under an iterated twist on the unit coball bundle over a P -manifold.

5.3.1. Lagrangian Floer homology on convex symplectic manifolds. We consider an exact compact connected symplectic manifold (M, ω) with boundary ∂M and two compact Lagrangian submanifolds L_0 and L_1 of M meeting the following hypotheses.

- (H1) L_0 and L_1 intersect transversally;
- (H2) $L_0 \cap L_1 \cap \partial M = \emptyset$;
- (H3) $H^1(L_j; \mathbb{R}) = 0$ for $j = 0, 1$.

We also assume that there exists a Liouville vector field X (i.e., $\mathcal{L}_X \omega = d\iota_X \omega = \omega$) which is defined on a neighbourhood U of ∂M and is everywhere transverse to ∂M , pointing outwards, such that

- (H4) $X(x) \in T_x L_j$ for all $x \in L_j \cap U$, $j = 0, 1$.

Let φ_r be the local semiflow of X defined near ∂M . Since ∂M is compact, we find $\epsilon > 0$ such that $\varphi_r(x)$ is defined for $x \in \partial M$ and $r \in [-\epsilon, 0]$. For these r we set

$$U_r = \bigcup_{r \leq r' \leq 0} \varphi_{r'}(\partial M).$$

In view of (H2) there exists $\epsilon' \in]0, \epsilon[$ such that for $V = U_{\epsilon'}$ we have

$$(15) \quad V \cap L_0 \cap L_1 = \emptyset.$$

An almost complex structure J on (M, ω) is called ω -compatible if $\omega \circ (\text{id} \times J)$ is a Riemannian metric on M . Following [10, 58, 7] we consider the space \mathcal{J} of smooth families $\mathbf{J} = \{J_t\}$, $t \in [0, 1]$, of smooth ω -compatible almost complex structures on M such that $J_t(x) = J(x)$ does not depend on t for $x \in V$ and such that

$$(J1) \quad \omega(X(x), J(x)v) = 0, \quad x \in \partial M, \quad v \in T_x \partial M,$$

$$(J2) \quad \omega(X(x), J(x)X(x)) = 1, \quad x \in \partial M,$$

$$(J3) \quad d\varphi_r(x)J(x) = J(\varphi_r(x))d\varphi_r(x), \quad x \in \partial M, \quad r \in [-\epsilon', 0].$$

For later use we examine conditions (J1) and (J2) closer. The contact structure ξ on ∂M is defined as

$$(16) \quad \xi = \{v \in T\partial M \mid \omega(X, v) = 0\},$$

and the Reeb vector field R on ∂M is defined by

$$(17) \quad \omega(X, R) = 1 \quad \text{and} \quad \omega(R, v) = 0 \quad \text{for all } v \in T\partial M.$$

Lemma 5.6. *Conditions (J1) and (J2) are equivalent to*

$$J\xi = \xi \quad \text{and} \quad JX = R.$$

The proof follows from definitions and the J -invariance of ω . It follows from Lemma 5.6 that the set \mathcal{J} is nonempty and connected, see [10]. Let

$$S = \{z = s + it \in \mathbb{C} \mid s \in \mathbb{R}, t \in [0, 1]\}$$

be the strip. The energy of $u \in C^\infty(S, M)$ is defined as

$$E(u) = \int_S u^* \omega.$$

For $u \in C^\infty(S, M)$ consider Floer's equation

$$(18) \quad \begin{cases} \partial_s u + J_t(u) \partial_t u = 0, \\ u(s, j) \in L_j, \quad j \in \{0, 1\}, \\ E(u) < \infty. \end{cases}$$

Notice that for a solution u of (18),

$$E(u) = \int_S \|\partial_s u\|^2 = \frac{1}{2} \int_S \|\partial_s u\|^2 + \|\partial_t u\|^2$$

is the energy of u associated with respect to any Riemannian metric defined via an ω -compatible J . It follows from (H1) that for every solution u of (18) there exist points $c_-, c_+ \in L_0 \cap L_1$ such that $\lim_{s \rightarrow \pm\infty} u(s, t) = c_\pm$ uniformly in t , cf. [43, Proposition 1.21]. The following lemma taken from [14, 31] shows that the images of solutions of (18) uniformly stay away from ∂M .

Lemma 5.7. *Let u be a finite energy solution of (18). Then*

$$u(S) \cap V = \emptyset.$$

Proof. Define $f: V \rightarrow \mathbb{R}$ by $f(\varphi_r(x)) = e^r$, where $x \in \partial M$ and $r \in [-\epsilon', 0]$. Using (J1), (J2), (J3) we find that the gradient ∇f with respect to each metric $\omega \circ (\text{id} \times J_t)$ is X ; for the function

$$F: \Omega = u^{-1}(V) \rightarrow \mathbb{R}, \quad (s, t) = z \mapsto F(z) = f \circ u(z),$$

one therefore computes $\Delta F = \langle \partial_s u, \partial_s u \rangle$, see e.g. [20], so that F is subharmonic. It follows that F does not attain a strict maximum on the interior of Ω . In order to see that this holds on Ω , fix a point $z \in \partial S$. We first assume $z = (s, 0)$, and claim that the function F satisfies the Neumann boundary condition at z ,

$$\partial_t F(z) = 0.$$

Indeed, we compute at z that

$$\begin{aligned} \partial_t F &= df(\partial_t u) = \langle \nabla f, \partial_t u \rangle = \langle X, \partial_t u \rangle \\ &= \omega(X, J\partial_t u) = -\omega(X, \partial_s u) = 0, \end{aligned}$$

where in the last step we have used that $X \in TL_0$ by (H4) and $\partial_s u \in TL_0$ by (18). Let now τ be the reflection $(s, t) \mapsto (s, -t)$, set $\widehat{\Omega} = \Omega \cup \tau(\Omega)$, and let \widehat{F} be the extension of F to $\widehat{\Omega}$ satisfying $\widehat{F}(s, -t) = F(s, t)$. Since $\partial_t F = 0$ along $\{t = 0\}$, the continuous function \widehat{F} is weakly subharmonic, and hence cannot have a strict maximum on $\widehat{\Omega}$. Repeating this argument for $z = (s, 1) \in \Omega$, we see that the same holds for F on Ω , and so either $u(S) \cap V = \emptyset$, or F is locally constant. In the latter case, $\Omega = S$, so that

$$\lim_{s \rightarrow \infty} u(s, t) = c_+ \in L_0 \cap V,$$

which is impossible in view of (15). \square

We endow \mathcal{J} with the C^∞ -topology. Recall that a subset of \mathcal{J} is *generic* if it is contained in a countable intersection of open and dense subsets. For $\mathbf{J} \in \mathcal{J}$ let $\mathcal{M}(\mathbf{J})$ be the space of solutions of (18). The following proposition is proved in [19, 37].

Proposition 5.8. *There exists a generic subset \mathcal{J}_{reg} of \mathcal{J} such that for each $\mathbf{J} \in \mathcal{J}_{\text{reg}}$ the moduli space $\mathcal{M}(\mathbf{J})$ is a smooth finite dimensional manifold.*

Under hypotheses (H1)–(H4), the ungraded Floer homology $HF(M, L_0, L_1)$ can be defined. In order to prove Theorem 5.4 we must compute the rank of this homology, and to this end it will be crucial to endow it with a \mathbb{Z} -grading. We therefore impose a final hypothesis. For $\mathbf{J} \in \mathcal{J}_{\text{reg}}$ denote the submanifold of those $u \in \mathcal{M}(\mathbf{J})$ with $\lim_{s \rightarrow \pm\infty} u(s, t) = c_\pm$ by $\mathcal{M}(c_-, c_+; \mathbf{J})$, and for $u \in \mathcal{M}(c_-, c_+; \mathbf{J})$ denote by $I(u)$ the local dimension of $\mathcal{M}(c_-, c_+; \mathbf{J})$ at u .

(H5) $I(u)$ only depends on c_- and c_+ .

Using (H5) and gluing one sees that there exists an index function

$$\text{ind}: L_0 \cap L_1 \rightarrow \mathbb{Z}$$

such that $I(u) = \text{ind } c_- - \text{ind } c_+$, so that

$$\dim \mathcal{M}(c_-, c_+; \mathbf{J}) = \text{ind } c_- - \text{ind } c_+.$$

For $k \in \mathbb{Z}$ let $CF_k(M, L_0, L_1)$ be the \mathbb{Z}_2 -vector space generated by the points $c \in L_0 \cap L_1$ with $\text{ind } c = k$. In view of (H1), the rank of $CF_k(M, L_0, L_1)$ is finite. In order to define a chain map on $CF_*(M, L_0, L_1)$ we need the following

Lemma 5.9. *For $u \in \mathcal{M}(c_-, c_+; \mathbf{J})$ the energy $E(u)$ only depends on c_- and c_+ .*

Proof. We have $E(u) = \int_u d\lambda = \int_{\partial u} \lambda = \int_{u(\mathbb{R}, 0)} \lambda - \int_{u(\mathbb{R}, 1)} \lambda$ for any primitive λ of ω . Since $d\lambda|_{L_j} = 0$ and $H^1(L_j; \mathbb{R}) = 0$, we find smooth functions f_i on L_j such that $\lambda|_{L_j} = df_j$ for $j = 0, 1$. Therefore, $E(u) = f_0(c_+) - f_0(c_-) - f_1(c_+) + f_1(c_-)$. \square

The group \mathbb{R} acts on $\mathcal{M}(c_-, c_+; \mathbf{J})$ by time-shift. In view of Lemma 5.7 the elements of $\mathcal{M}(c_-, c_+; \mathbf{J})$ uniformly stay away from the boundary ∂M , and by Lemma 5.9 and (H1), their energy is uniformly bounded. Moreover, $[\omega]|_{\pi_2(M)} = 0$ and $[\omega]|_{\pi_2(M, L_j)} = 0$ since ω is exact and by (H3), so that when taking limits in $\mathcal{M}(c_-, c_+; \mathbf{J})$ there is no bubbling off of \mathbf{J} -holomorphic spheres or discs. The Floer-Gromov compactness theorem thus implies that the quotient $\mathcal{M}(c_-, c_+; \mathbf{J})/\mathbb{R}$ is compact. In particular, if $\text{ind } c_- - \text{ind } c_+ = 1$, then $\mathcal{M}(c_-, c_+; \mathbf{J})/\mathbb{R}$ is a finite set, and we then set

$$n(c_-, c_+; \mathbf{J}) = \# \{ \mathcal{M}(c_-, c_+; \mathbf{J})/\mathbb{R} \} \pmod{2}.$$

For $k \in \mathbb{Z}$ define the Floer boundary operator $\partial_k(\mathbf{J}): CF_k \rightarrow CF_{k-1}$ as the linear extension of

$$\partial_k(\mathbf{J})c = \sum_{\substack{c' \in L_0 \cap L_1 \\ i(c')=k-1}} n(c', c) c'.$$

Using the compactness of the 0- and 1-dimensional parts of $\mathcal{M}(\mathbf{J})/\mathbb{R}$ one shows by gluing that $\partial_{k-1}(\mathbf{J}) \circ \partial_k(\mathbf{J}) = 0$ for each k , see [16, 44]. The complex $(CF_*(M, L_0, L_1; \mathbf{J}), \partial_*(\mathbf{J}))$ is called the Floer chain complex. A continuation argument together with Lemma 5.7 shows that its homology

$$HF_k(M, L_0, L_1; \mathbf{J}) = \frac{\ker \partial_k(\mathbf{J})}{\text{im } \partial_{k+1}(\mathbf{J})}$$

is a graded \mathbb{Z}_2 -vector space which does not depend on $\mathbf{J} \in \mathcal{J}_{\text{reg}}$, see again [16, 44], and so we can define the Lagrangian Floer homology of

the triple (M, L_0, L_1) by

$$HF_*(M, L_0, L_1) = HF_*(M, L_0, L_1; \mathbf{J})$$

for any $\mathbf{J} \in \mathcal{J}_{\text{reg}}$. We denote by $\text{Ham}^c(M) \subset \text{Symp}^c(M)$ the group of Hamiltonian diffeomorphisms generated by time-dependent Hamiltonian functions whose support is contained in $\text{Int } M$. The usual continuation argument also implies

Proposition 5.10. *For any $\varphi \in \text{Ham}^c(M)$ we have $HF_*(M, \varphi(L_0), L_1) = HF_*(M, L_0, L_1)$ as graded \mathbb{Z}_2 -vector spaces.*

5.4. Computation of $HF_*(\vartheta^m(D_x), D_y)$. We consider a d -dimensional P_k -manifold (B, g) . Using the Riemannian metric g we identify T_1^*B with the unit ball bundle T_1B , and for $x \in B$ we set $D_x = T_xB \cap T_1B$. According to Lemma 5.2 we find $x \in B$ such that $V_x = \{v \in \partial D_x \mid l(v) = 1\}$ is a non-empty and open subset of ∂D_x . We denote by ρ the injectivity radius at x , and we define the non-empty open subset W of B by

$$W = \exp_x \left\{ v \in T_xB \mid 0 < |v| < \rho, \frac{v}{|v|} \in V_x \right\}.$$

Let $f: [0, \infty[\rightarrow [0, \infty[$ be a smooth function as in (11). More precisely, we choose f such that

$$f(r) = 0 \text{ if } r \in [0, \frac{1}{3}], \quad f'(r) = 1 \text{ if } r \geq \frac{2}{3}, \quad f''(r) > 0 \text{ if } r \in]\frac{1}{3}, \frac{2}{3}[.$$

Fix $m \in \mathbb{Z} \setminus \{0\}$. For notational convenience we assume $m \geq 1$. The symplectomorphism $\vartheta^m = \vartheta_f^m \in \text{Symp}^c(T^*B)$ is generated by $mf(|p|)$. Choose now $y \in W$. By our choice of f the two Lagrangian submanifolds

$$L_0 = \vartheta^m(D_x) \quad \text{and} \quad L_1 = D_y$$

intersect transversely in exactly $2m$ points and in particular meet hypothesis (H1); moreover, ϑ^m is the identity on $U = T_1^*B \setminus T_{2/3}^*B$, so that $L_0 \cap L_1 \cap U = \emptyset$ and (H2) is met. Since L_0 and L_1 are simply connected, (H3) is also met, and

$$(19) \quad X = X(q, p) = \sum_{i=1}^d p_i \frac{\partial}{\partial p_i}$$

is a Liouville vector field defined on all of T_1^*B which is transverse to ∂T_1^*B , pointing outwards, and $X(x) \in T_xL_j$ for all $x \in L_j \cap U$, $j = 0, 1$, verifying (H4). In order to verify (H5) we follow [46] and describe the natural grading on $HF(T_1^*B, L_0, L_1)$. Let δ be the distance of y from x ; then $0 < \delta < \rho < 1/2$. For $i \in \mathbb{N}_m = \{0, 1, \dots, m-1\}$ we set

$$\tau_i^+ = i + \delta \quad \text{and} \quad \tau_i^- = i + 1 - \delta$$

and define $r_i^\pm \in]\frac{1}{3}, \frac{2}{3}[$ by $mf'(r_i^\pm) = \tau_i^\pm$. The $2m$ points in $L_0 \cap L_1$ are then given by

$$c_i^\pm = \vartheta^m(r_i^\pm \dot{\gamma}^\pm(0)) = r_i^\pm \dot{\gamma}^\pm(\delta)$$

where $\gamma^+ : \mathbb{R} \rightarrow B$ is the geodesic with $\gamma^+(0) = x$ and $\gamma(\delta) = y$ and $\gamma^-(t) = \gamma^+(-t)$ is the opposite geodesic, cf. Figure 3.

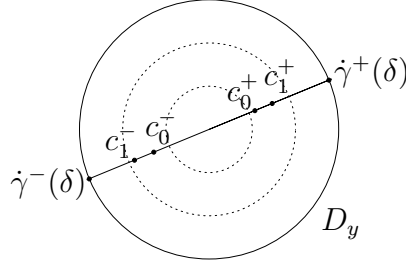


FIGURE 3. The points $c_i^\pm \in L_0 \cap L_1$ for $m = 2$.

Define the index function $\text{ind} : L_0 \cap L_1 \rightarrow \mathbb{Z}$ by

$$\text{ind } c_i^\pm = \sum_{0 < t < \tau_i^\pm} \text{ind } \gamma^\pm(t), \quad i \in \mathbb{N}_m.$$

It is shown in [46] that for $\mathbf{J} \in \mathcal{J}_{\text{reg}}$ and $u \in \mathcal{M}(c_-, c_+; \mathbf{J})$ the local dimension $I(u)$ of $\mathcal{M}(c_-, c_+; \mathbf{J})$ at u is $\text{ind } c_- - \text{ind } c_+$, so that (H5) is also met. We abbreviate

$$CF_*(B, m) = CF_*(T_1^*B, L_0, L_1) \quad \text{and} \quad HF_*(B, m) = HF_*(T_1^*B, L_0, L_1).$$

Our next goal is to compute the Floer chain groups $CF_*(B, m)$.

Proposition 5.11. *Let (B, g) be a d -dimensional P_k -manifold and $i \in \mathbb{N}_m$. Then*

$$\text{ind } c_i^+ = i(k + d - 1) \quad \text{and} \quad \text{ind } c_i^- = i(k + d - 1) + k.$$

Proof. We start with a general

Lemma 5.12. *Let $\gamma : \mathbb{R} \rightarrow B$ be a geodesic of a P -manifold (B, g) , and let $J : \mathbb{R} \rightarrow \mathbb{R}$ be a Jacobi field along γ such that $J(0) = 0$. If $J(t) = 0$, then $J(t + n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. We can assume that $J'(t) \neq 0$ since otherwise $J \equiv 0$. Fix $n \in \mathbb{Z}$. Since (B, g) is a P -manifold, $\gamma(t + n)$ is conjugate to $\gamma(t)$ with multiplicity $d - 1$. Since this is the maximal possible multiplicity of a conjugate point, and since $J(t) = 0$ and $J'(t) \neq 0$, J must be a Jacobi field conjugating $J(t)$ and $J(t + n)$, i.e., $J(t + n) = 0$. \square

By our choice of y , the point y is not conjugate to x along $\gamma^+ : [0, \delta] \rightarrow B$, and so $\text{ind } c_0^+ = 0$. Since (B, g) is a d -dimensional P_k -manifold, Lemma 5.12 now implies $\text{ind } c_i^+ = i(k+d-1)$ for all $i \in \mathbb{N}_m$. The choice of y and Lemma 5.12 imply that $\text{ind } c_0^- = k$, and now Lemma 5.12 implies $\text{ind } c_i^- = i(k+d-1) + k$ for all $i \in \mathbb{N}_m$. \square

In view of Proposition 5.11 we find

Corollary 5.13. *Let (B, g) be a d -dimensional P_k -manifold.*

If $k \geq 1$,

$$CF_i(B, m) = \begin{cases} \mathbb{Z}_2 & i \in (k+d-1)\mathbb{N}_m \cup ((k+d-1)\mathbb{N}_m + k) \\ 0 & \text{otherwise;} \end{cases}$$

if $k = 0$ and $d > 1$,

$$CF_i(B, m) = \begin{cases} \mathbb{Z}_2^2 & i \in (d-1)\mathbb{N}_m \\ 0 & \text{otherwise;} \end{cases}$$

if $k = 0$ and $d = 1$,

$$CF_i(B, m) = \begin{cases} \mathbb{Z}_2^{2m} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.14. *Assume that (B, g) is a P_k -manifold as in Theorem 5.4, and if $k = 1$ assume that $(B, g) = (\mathbb{CP}^n, g_{\text{can}})$. Then the Floer boundary operator $\partial_* : CF_*(B, m) \rightarrow CF_{*-1}(B, m)$ vanishes identically, and so $HF_*(B, m) = CF_*(B, m)$. In particular, $\text{rank } HF(B, m) = 2m$.*

Proof. We first assume that $k \neq 1$ and $d \neq 2$. Corollary 5.13 then shows that for any $*$ $\in \mathbb{Z}$, at least one of the chain groups $CF_*(B, m)$ and $CF_{*-1}(B, m)$ is trivial, and so $\partial_* = 0$. It remains to prove the vanishing of ∂_* for the spaces \mathbb{RP}^2 and \mathbb{CP}^n , $n \geq 1$, endowed with their canonical P -metrics. We shall do this by using a symmetry argument.

The case $(\mathbb{CP}^n, g_{\text{can}})$. Note that every diffeomorphism φ of \mathbb{CP}^n lifts to a symplectomorphism $(\varphi^{-1})^*$ of $T^*\mathbb{CP}^n$, and that if φ is an isometry, then $(\varphi^{-1})^*$ is a symplectomorphism of $T_1^*\mathbb{CP}^n$. Let \mathbb{RP}^n be the real locus of \mathbb{CP}^n .

Lemma 5.15. *We can assume without loss of generality that $x, y \in \mathbb{RP}^n$.*

Proof. Choose a unitary matrix $U \in \mathbf{U}(n+1)$ such that $x' = U(x) = [1 : 0 : \dots : 0] \in \mathbb{RP}^n$ and $y' = U(y) \in \mathbb{RP}^n$. We again identify $T_1^*\mathbb{CP}^n$ with $T_1\mathbb{CP}^n$ via the Riemannian metric g_{can} . Since U is an isometry

of $(\mathbb{CP}^n, g_{\text{can}})$, its lift U_* to $T_1 \mathbb{CP}^n$ commutes with the geodesic flow on $T_1 \mathbb{CP}^n$, and hence $(U^{-1})^*$ commutes with ϑ^m . Therefore,

$$(U^{-1})^* L_0 = (U^{-1})^* \vartheta^m(D_x) = \vartheta^m(U^{-1})^*(D_x) = \vartheta^m(D_{x'})$$

and $(U^{-1})^* L_1 = (U^{-1})^* D_y = D_{y'}$. By the natural invariance of Lagrangian Floer homology we thus obtain

$$\begin{aligned} HF_*(T_1 \mathbb{CP}^n, L_0, L_1) &= HF_*(T_1 \mathbb{CP}^n, (U^{-1})^* L_0, (U^{-1})^* L_1) \\ &= HF_*(T_1 \mathbb{CP}^n, \vartheta^m(D_{x'}), D_{y'}), \end{aligned}$$

as desired. \square

Consider the involution

$$(20) \quad [z_0 : z_1 : \cdots : z_n] \mapsto [\bar{z}_0 : \bar{z}_1 : \cdots : \bar{z}_n]$$

of \mathbb{CP}^n . Its fixed point set is \mathbb{RP}^n . Since complex conjugation (20) is an isometry of $(\mathbb{CP}^n, g_{\text{can}})$, it lifts to a symplectic involution σ of $T_1^* \mathbb{CP}^n$. Since $x, y \in \mathbb{RP}^n$ and since complex conjugation is an isometry, we see as in the proof of Lemma 5.15 that $\sigma(L_j) = L_j$, $j = 0, 1$, and σ acts trivially on $L_0 \cap L_1$. Assume that $\mathbf{J} \in \mathcal{J}(T_1^* \mathbb{CP}^n)$ is invariant under σ , i.e., $\sigma^* J_t = \sigma_* J_t \sigma_* = J_t$ for every $t \in [0, 1]$. Then σ induces an involution on the solutions of (18) by

$$u \mapsto \sigma \circ u.$$

If u is invariant under σ , i.e., $u = \sigma \circ u$, then u is a solution of (18) with M replaced by the fixed point set $M^\sigma = T_1^* \mathbb{RP}^n$ of σ and L_j replaced by $L_j^\sigma = L_j \cap M^\sigma$ for $j = 0, 1$. According to Proposition 5.1, \mathbb{CP}^n is a P_1 -manifold and \mathbb{RP}^n is a P_0 -manifold, and so we read off from Corollary 5.13 that if $\text{ind}_M(c_-) - \text{ind}_M(c_+) = 1$, then $\text{ind}_{M^\sigma}(c_-) - \text{ind}_{M^\sigma}(c_+) = 0$. One thus expects that for generic σ -invariant $\mathbf{J} \in \mathcal{J}$ there are no solutions of (18) which are invariant under σ . In particular, solutions of (18) appear in pairs, and so $\partial_* = 0$. To make this argument precise, we need to show that there exist σ -invariant $\mathbf{J} \in \mathcal{J}$ which are “regular” for every non-invariant solution of (18) and whose restriction to M^σ is also “regular”. This will be done in the next paragraph.

5.4.1. A transversality theorem. We consider, more generally, an exact compact symplectic manifold (M, ω) with boundary ∂M containing two compact Lagrangian submanifolds L_0 and L_1 as in 5.3.1: (H1), (H2), (H3) hold and there is a Liouville vector field X on a neighbourhood U of ∂M such that (H4) holds. We in addition assume that σ is a symplectic involution of (M, ω) such that

$$(21) \quad \sigma(L_j) = L_j, \quad j = 0, 1, \quad \sigma|_{L_0 \cap L_1} = \text{id}, \quad \sigma_* X = X.$$

We have already verified the first two properties for $M = T_1^* \mathbb{CP}^n$ and the lift σ of (20), and we notice that $\sigma_* X = X$ for the Liouville vector field (19). The fixed point set $M^\sigma = \text{Fix}(\sigma)$ is a symplectic submanifold of (M, ω) . Set $\omega^\sigma = \omega|_{M^\sigma}$. Since $\sigma_* X = X$ the vector field $X^\sigma = X|_{U \cap M^\sigma}$ is a Liouville vector field near ∂M^σ . As in 5.3.1 we denote by $\mathcal{J} = \mathcal{J}(M)$ the space of smooth families $\mathbf{J} = \{J_t\}$, $t \in [0, 1]$, of smooth ω -compatible almost complex structures on M which on V do not depend on t and meet (J1), (J2), (J3). The space $\mathcal{J}(M^\sigma)$ is defined analogously by imposing (J1), (J2), (J3) for X^σ on $M^\sigma \cap V$. The subspace of those \mathbf{J} in $\mathcal{J}(M)$ which are σ -invariant is denoted $\mathcal{J}^\sigma(M)$. There is a natural restriction map

$$\varrho: \mathcal{J}^\sigma(M) \rightarrow \mathcal{J}(M^\sigma), \quad \mathbf{J} \mapsto \mathbf{J}|_{TM^\sigma}.$$

Lemma 5.16. *The restriction map ϱ is open.*

Proof. Recall that φ_r , $r \leq 0$, denotes the semiflow of X , and that ξ and R are the contact structure and the Reeb vector field on ∂M defined by (16) and (17). Since σ is symplectic, $\sigma_* X = X$ and $\sigma(\partial M) = \partial M$ we have

$$\sigma_* \xi = \xi \quad \text{and} \quad \sigma_* R = R.$$

The contact structure ξ^σ on ∂M^σ associated with X^σ is $\xi \cap T\partial M^\sigma$, and the Reeb vector field R^σ is $R|_{\partial M^\sigma}$. We shall prove Lemma 5.16 by first showing that ρ is onto. From the proof it will then easily follow that ρ is open.

Step 1. ρ is onto: Fix $\mathbf{J}^\sigma \in \mathcal{J}(M^\sigma)$. We set $g_t^\sigma = \omega \circ (\text{id} \times J_t^\sigma)$. Choose a smooth family $\mathbf{g} = \{g_t\}$, $t \in [0, 1]$, of Riemannian metrics on TM which on V does not depend on t and satisfies

- (g1) $g(X(x), v) = 0$, $x \in \partial M$, $v \in T_x \partial M$,
- (g2) $g(X(x), X(x)) = 1$, $x \in \partial M$,
- (g3) $\varphi_r^* g(x) = e^r g(x)$, $x \in \partial M$, $r \in [-\epsilon', 0]$,
- (g4) $g(R(x), R(x)) = 1$, $x \in \partial M$,
- (g5) $g(R(x), v) = 0$, $x \in \partial M$, $v \in \xi$,

and in addition satisfies for each t

- (g6) if $x \in M^\sigma$, then $g_t(x)|_{T_x M^\sigma} = g_t^\sigma(x)$,
- (g7) if $x \in M^\sigma$, then the Riemannian and the symplectic orthogonal complement of $T_x M^\sigma$ in $T_x M$ coincide, i.e., if for $\eta \in T_x M$ it holds that $\omega(\eta, \zeta) = 0$ for every $\zeta \in T_x M^\sigma$, then also $g(\eta, \zeta) = 0$ for every $\zeta \in T_x M^\sigma$,
- (g8) $\sigma^* g_t = g_t$.

In order to see that such a family \mathbf{g} exists, first notice that in view of (J1), (J2), (J3), Lemma 5.6 and (17), the metric g^σ satisfies (g1)–(g5) for X^σ , R^σ , $x \in \partial M^\sigma$ and $v \in T_x \partial M^\sigma$ or $v \in \xi^\sigma$. We thus find a family \mathbf{g}_0 satisfying (g1)–(g7). Then $\sigma^* \mathbf{g}_0$ also satisfies (g1)–(g7) as one readily verifies; we only mention that (g3) follows from $\sigma^* \circ \varphi_r = \varphi_r \circ \sigma$ which is a consequence of $\sigma_* X = X$. Now set

$$\mathbf{g} = \frac{1}{2} (\mathbf{g}_0 + \sigma^* \mathbf{g}_0).$$

Let \mathfrak{Met} be the space of smooth Riemannian metrics on M and let $\mathcal{J}(\omega)$ be the space of smooth ω -compatible almost complex structures on M . For $J \in \mathcal{J}(\omega)$ we write $g_J = \omega \circ (\text{id} \times J) \in \mathfrak{Met}$. It is shown in [33, Proposition 2.50 (ii)] that there exists a smooth map

$$r: \mathfrak{Met} \rightarrow \mathcal{J}(\omega), \quad g \mapsto r(g) =: J_g$$

such that

$$(22) \quad r(g_J) = J \quad \text{and} \quad r(\varphi^* g) = \varphi^* r(g)$$

for every symplectomorphism φ of M . We define $\mathbf{J} = \{J_t\}$ by

$$J_t = r(g_t).$$

The second property in (22) and (g8) show that \mathbf{J} is σ -invariant. In order to prove that $\mathbf{J} \in \mathcal{J}^\sigma(M)$ we also need to show that each J_t meets (J1), (J2), (J3) and $J_t|_{M^\sigma} = J_t^\sigma$. To this end we must recall the construction of r from [33]. Fix $g \in \mathfrak{Met}$ and $x \in M$. The automorphism A of $T_x M$ defined by $\omega_x(v, w) = g_x(Av, w)$ is g -skew-adjoint. Denoting by A^* its g -adjoint, we find that $P = A^* A = -A^2$ is g -positive definite. Let Q be the automorphism of $T_x M$ which is g -self-adjoint, g -positive definite, and satisfies $Q^2 = P$, and then set

$$J_x(\omega, g) = Q^{-1} A.$$

It is clear that $J_x(\omega, g)$ depends smoothly on x . The map r is defined by $r(g)(x) = J_x(\omega, g)$. One readily verifies that $r(g)$ is ω -compatible, see [27, p. 14], and meets (22). From the construction we moreover read off that

- (r1) $J_x(c_1 \omega, c_2 g) = J_x(\omega, g)$ for all $c_1, c_2 > 0$,
- (r2) if $T_x M = V \oplus W$ in such a way that W is both ω -orthogonal and g -orthogonal to V , i.e., $W = V^\omega = V^\perp$, then A , P and Q leave both V and W invariant, so that $J_x(\omega, g)$ leaves V invariant and

$$J_x(\omega, g)|_V = J_x(\omega|_V, g|_V).$$

We are now in position to verify (J1), (J2), (J3) for $J_t = r(g_t) = J_{g_t}$. In view of (16) and (17) and (g1) and (g5) the plane field $\langle X, R \rangle$ on ∂M generated by X and R is both ω -orthogonal and g -orthogonal to ξ ,

$$\langle X, R \rangle = \xi^\omega = \xi^\perp,$$

and so (r2) implies

$$(23) \quad J_g|_{\langle X, R \rangle} = J_{g|_{\langle X, R \rangle}}.$$

Define the complex structure J_0 on $\langle X, R \rangle$ by $J_0 X = R$. Using (g1), (g2), (g4) and (17) we find $g|_{\langle X, R \rangle} = g_{J_0}$, and so the first property in (22) implies $J_{g|_{\langle X, R \rangle}} = J_0$. Together with (23) we find

$$(24) \quad J_g|_{\langle X, R \rangle} = J_0.$$

The J_g -invariance of ω , (24) and (17) yield (J1) and (J2). The identity (J3) follows from $\varphi_r^* \omega = e^r \omega$, (g3) and (r1). Finally, $J_t|_{M^\sigma} = J_t^\sigma$ follows from (g6), (g7), (r2) and the first property in (22).

Step 2. ρ is open: Let U be an open subset of $\mathcal{J}^\sigma(M)$. We must show that given $\mathbf{J}^\sigma \in \rho(U)$, every (C^∞) -close enough $\tilde{\mathbf{J}}^\sigma \in \mathcal{J}(M^\sigma)$ belongs to $\rho(U)$. Fix $\mathbf{J} \in U$ with $\rho(\mathbf{J}) = \mathbf{J}^\sigma$, and set $\mathbf{g} = g_{\mathbf{J}}$. Since $\mathbf{J} \in \mathcal{J}^\sigma(M)$, the family \mathbf{g} satisfies (g1)–(g8). If $\tilde{\mathbf{J}}^\sigma \in \mathcal{J}(M^\sigma)$ is close to \mathbf{J}^σ , then $\tilde{\mathbf{g}}^\sigma = g_{\tilde{\mathbf{J}}^\sigma}$ is close to $g_{\mathbf{J}^\sigma}$, and so we can choose a smooth family $\tilde{\mathbf{g}}_0$ close to \mathbf{g} which satisfies (g1)–(g7). Then

$$\tilde{\mathbf{g}} = \frac{1}{2}(\tilde{\mathbf{g}}_0 + \sigma^* \tilde{\mathbf{g}}_0)$$

satisfies (g1)–(g8), and since $\tilde{\mathbf{g}}_0$ was close to \mathbf{g} and since $\sigma^* \mathbf{g} = \mathbf{g}$, the family $\tilde{\mathbf{g}}$ is also close to \mathbf{g} . Set $\tilde{\mathbf{J}} = r(\tilde{\mathbf{g}})$. Then $\rho(\tilde{\mathbf{J}}) = \tilde{\mathbf{J}}^\sigma$, and since $r: \mathcal{M} \rightarrow \mathcal{J}(\omega)$ is smooth and $\tilde{\mathbf{g}}$ is close to \mathbf{g} , we see that $\tilde{\mathbf{J}} = r(\tilde{\mathbf{g}})$ is close to $r(\mathbf{g}) = r(g_{\mathbf{J}}) = \mathbf{J}$. In particular, if $\tilde{\mathbf{J}}^\sigma$ was close enough to \mathbf{J}^σ , then $\tilde{\mathbf{J}} \in U$. The proof of Lemma 5.16 is complete. \square

For the remainder of the proof of Theorem 5.14 for $(\mathbb{CP}^n, g_{\text{can}})$ we assume that the reader is familiar with the standard transversality arguments in Floer theory as presented in Section 5 of [19] or Sections 3.1 and 3.2 of [34], and we shall focus on those aspects of the argument particular to our situation. Fix $c_-, c_+ \in L_0 \cap L_1$. We interpret solutions of (18) with $\lim_{s \rightarrow \pm\infty} u(s, t) = c_\pm$ as the zero set of a smooth section from a Banach manifold \mathcal{B} to a Banach bundle \mathcal{E} over \mathcal{B} . We fix $p > 2$. According to Lemma D.1 in [42] there exists a smooth family of Riemannian metrics $\{g_t\}$, $t \in [0, 1]$, on M such that L_j is totally geodesic with respect to g_j , $j = 0, 1$. Let $\mathcal{B} = \mathcal{B}^{1,p}(c_-, c_+)$ be the space of continuous maps u from the strip $S = \mathbb{R} \times [0, 1]$ to the interior $\text{Int } M$

of M which satisfy $\lim_{s \rightarrow \pm\infty} u(s, t) = c_{\pm}$ uniformly in t , are locally of class $W^{1,p}$, and satisfy the conditions

- (B1) $u(s, j) \in L_j$ for $j = 0, 1$,
- (B2) there exists $T > 0$, $\xi_- \in W^{1,p}((-\infty, -T] \times [0, 1], T_{c_-}M)$, and $\xi_+ \in W^{1,p}([T, \infty) \times [0, 1], T_{c_+}M)$ with $\xi_{\pm}(s, j) \in T_{c_{\pm}}L_j$ such that

$$u(s, t) = \begin{cases} \exp_{c_-}(\xi_-(s, t)), & s \leq -T, \\ \exp_{c_+}(\xi_+(s, t)), & s \geq T. \end{cases}$$

Here, $\exp_{c_{\pm}}(\xi_{\pm}(s, t))$ denotes the image of $\xi_{\pm}(s, t)$ under the exponential map with respect to g_t at c_{\pm} . The space \mathcal{B} is an infinite dimensional Banach manifold whose tangent space at u is

$$T_u\mathcal{B} = \{\xi \in W^{1,p}(S, u^*TM) \mid \xi(s, j) \in T_{u(s,j)}L_j, j = 0, 1\}.$$

Let \mathcal{E} be the Banach bundle over \mathcal{B} whose fibre over $u \in \mathcal{B}$ is

$$\mathcal{E}_u = L^p(S, u^*TM).$$

For $\mathbf{J} \in \mathcal{J}(M)$ define the section $\mathcal{F}_{\mathbf{J}}: \mathcal{B} \rightarrow \mathcal{E}$ by

$$\mathcal{F}_{\mathbf{J}}(u) = \partial_s u + J_t(u) \partial_t(u)$$

and set $\mathcal{M}_{\mathbf{J}} = \mathcal{F}_{\mathbf{J}}^{-1}(0)$. The set $\mathcal{M}_{\mathbf{J}}$ agrees with the set of those $u \in \mathcal{M}(\mathbf{J})$ with $\lim_{s \rightarrow \pm\infty} u(s, t) = c_{\pm}$. Indeed, Lemma 5.7 and Proposition 1.21 in [19] show that the latter set belongs to $\mathcal{M}_{\mathbf{J}}$. Conversely, in view of $p > 2$, elliptic regularity and (B2) imply that $u \in \mathcal{M}_{\mathbf{J}}$ is smooth and satisfies $\lim_{s \rightarrow \pm\infty} \partial_s u(s, t) = 0$ uniformly in t , so that $E(u) < \infty$ by Proposition 1.21 in [19]. If $u \in \mathcal{M}_{\mathbf{J}}$, then the vertical differential of $\mathcal{F}_{\mathbf{J}}$,

$$D_{u,\mathbf{J}} \equiv D\mathcal{F}_{\mathbf{J}}(u): T_u\mathcal{B} \rightarrow \mathcal{E}_u, \quad \xi \mapsto \nabla_s \xi + J_t(u) \nabla_t \xi + \nabla_{\xi} J_t(u) \partial_t u,$$

is a Fredholm operator, cf. [43, Theorem 2.2]. Here, ∇ denotes the Levi-Civita connection with respect to the t -dependent metric g_t . We further consider the Banach submanifold

$$\mathcal{B}^{\sigma} = \{u \in \mathcal{B} \mid u = \sigma \circ u\},$$

of those u in \mathcal{B} whose image lies in M^{σ} . We denote by \mathcal{E}^{σ} the Banach bundle over \mathcal{B}^{σ} whose fibre over $u \in \mathcal{B}^{\sigma}$ is

$$\mathcal{E}_u^{\sigma} = L^p(S, u^*TM^{\sigma}).$$

Note that \mathcal{E}^{σ} is a subbundle of the restriction of \mathcal{E} to \mathcal{B}^{σ} . For $\mathbf{J} \in \mathcal{J}^{\sigma}(M)$ we abbreviate

$$\mathcal{M}_{\mathbf{J}}^{\sigma} \equiv \mathcal{F}_{\mathbf{J}}^{-1}(0) \cap \mathcal{B}^{\sigma} = \mathcal{M}_{\mathbf{J}} \cap \mathcal{B}^{\sigma}$$

and for $u \in \mathcal{M}_{\mathbf{J}}^\sigma$ we set

$$D_{u,\mathbf{J}}^\sigma \equiv D_{u,\mathbf{J}}|_{T_u\mathcal{B}^\sigma} : T_u\mathcal{B}^\sigma \rightarrow \mathcal{E}_u^\sigma.$$

Definition 5.17. We say that $\mathbf{J} \in \mathcal{J}^\sigma(M)$ is *regular* if for every $u \in \mathcal{M}_{\mathbf{J}} \setminus \mathcal{M}_{\mathbf{J}}^\sigma$ the operator $D_{u,\mathbf{J}}$ is onto and if for every $u \in \mathcal{M}_{\mathbf{J}}^\sigma$ the operator $D_{u,\mathbf{J}}^\sigma$ is onto.

Proposition 5.18. *The set $(\mathcal{J}^\sigma(M))_{\text{reg}}$ of regular almost complex structures is generic in $\mathcal{J}^\sigma(M)$.*

Proof. It is proved in [31, Proposition 5.13] that the subset \mathcal{R}_1 of those $\mathbf{J} \in \mathcal{J}^\sigma(M)$ for which $D_{u,\mathbf{J}}$ is onto for every $u \in \mathcal{M}_{\mathbf{J}} \setminus \mathcal{M}_{\mathbf{J}}^\sigma$ is generic in $\mathcal{J}^\sigma(M)$. Moreover, it is proved in [19, Section 5] that the subset \mathcal{R}_2^σ of those $\mathbf{J}^\sigma \in \mathcal{J}(M^\sigma)$ for which D_{u,\mathbf{J}^σ} is onto for every $u \in \mathcal{M}_{\mathbf{J}^\sigma}^\sigma$ is generic in $\mathcal{J}(M^\sigma)$. Notice that for $\mathbf{J} \in \mathcal{J}^\sigma(M)$ we have $\mathcal{M}_{\mathbf{J}}^\sigma = \mathcal{M}_{\rho(\mathbf{J})}$ and $D_{u,\mathbf{J}}^\sigma = D_{u,\rho(\mathbf{J})}$ for $u \in \mathcal{M}_{\mathbf{J}}^\sigma = \mathcal{M}_{\rho(\mathbf{J})}$. It follows that for $\mathbf{J} \in \mathcal{R}_2 \equiv \rho^{-1}(\mathcal{R}_2^\sigma)$ the operator $D_{u,\mathbf{J}}^\sigma$ is onto for every $u \in \mathcal{M}_{\mathbf{J}^\sigma}$. Since the preimage of a generic set under a continuous open map is generic, \mathcal{R}_2 is generic in $\mathcal{J}^\sigma(M)$. Therefore, the set of regular $\mathbf{J} \in \mathcal{J}^\sigma(M)$ contains the generic set $\mathcal{R}_1 \cap \mathcal{R}_2$, and the proof of Proposition 5.18 is complete. \square

In order to complete the proof of Theorem 5.14 for $(\mathbb{CP}^n, g_{\text{can}})$, set again $M = T_1^* \mathbb{CP}^n$. In view of Proposition 5.18 we find a $\mathbf{J} \in \mathcal{J}^\sigma(M)$ which is regular for all $c_-, c_+ \in L_0 \cap L_1$. Fix c_-, c_+ with $\text{ind}_M(c_-) - \text{ind}_M(c_+) = 1$. Since $\text{ind}_{M^\sigma}(c_-) - \text{ind}_{M^\sigma}(c_+) = 0$, the Fredholm index of $D_{u,\mathbf{J}}^\sigma$ for $u \in \mathcal{M}_{\mathbf{J}}^\sigma$ vanishes, so that the manifold of solutions of (18) contained in M^σ is 0-dimensional and hence empty. Moreover, $D_{u,\mathbf{J}}$ is onto for every $u \in \mathcal{M}_{\mathbf{J}} \setminus \mathcal{M}_{\mathbf{J}}^\sigma$, and so $D_{u,\mathbf{J}}$ is onto for every $u \in \mathcal{M}_{\mathbf{J}}$. We can thus compute the Floer homology $HF_*(M, L_0, L_1)$ by using \mathbf{J} . Since $\mathcal{M}_{\mathbf{J}}^\sigma$ is empty, $\partial_* = 0$, and the proof of Theorem 5.14 for $(\mathbb{CP}^n, g_{\text{can}})$ is complete.

The case $(\mathbb{RP}^2, g_{\text{can}})$. We consider the involution

$$(25) \quad [q_0 : q_1 : q_2] \mapsto [-q_0 : q_1 : q_2]$$

of \mathbb{RP}^2 . Its fixed point set is $[1 : 0 : 0] \cup \{[0 : q_1 : q_2]\} \equiv p_N \cup \mathbb{RP}^1 = p_N \cup S^1$. Since every isometry of (S^2, g_{can}) descends to an isometry of $(\mathbb{RP}^2, g_{\text{can}})$, we find an isometry of \mathbb{RP}^2 mapping x and y to S^1 ; by the argument in the proof of Lemma 5.15 we can thus assume that $x, y \in S^1$. Let σ be the symplectic involution of $T_1^* \mathbb{RP}^2$ obtained from lifting (25). It satisfies (21). The fixed point set of σ is $p_N \cup T_1^* S^1$. Since p_N is disjoint from $L_0 \cap L_1$, there is no solution of (18) lying in p_N . Set $M^\sigma = T_1^* S^1$. According to Proposition 5.1,

both \mathbb{RP}^2 and S^1 are P_0 -manifolds, and so Corollary 5.13 implies that if $\text{ind}_M(c_-) - \text{ind}_M(c_+) = 1$, then $\text{ind}_{M^\sigma}(c_-) - \text{ind}_{M^\sigma}(c_+) = 0$. The vanishing of ∂_* now follows as in 5.4.1. The proof of Theorem 5.14 is finally complete. \square

5.5. End of the proof. For $(B, g) = S^1$, Theorem 5.4 follows from the topological argument given in Section 2, see also Corollary 5.21 below. For the remainder of this subsection we therefore assume that (B, g) is a P -manifold of dimension $d \geq 2$. We abbreviate $M = T^*B$ and $M_r = T_r^*B$.

Lemma 5.19. $\text{Ham}^c(M) = \text{Symp}_0^c(M)$.

Proof. Since T^*B is orientable, Poincaré duality yields

$$H_c^1(M; \mathbb{R}) \cong H_{2d-1}(M; \mathbb{R}) \cong H_{2d-1}(B; \mathbb{R}) = 0.$$

The lemma now follows in view of the exact sequence (6). \square

Let now (B, g) be a P -manifold as in Theorem 5.14. Let $\vartheta = \vartheta_f$ be the twist considered in Subsection 5.4, and let $\varphi \in \text{Symp}^c(M)$ be such that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(M))$ for some $m \in \mathbb{Z} \setminus \{0\}$. We assume without loss of generality that $m \geq 1$. By Lemma 5.19 we find $r > 0$ such that $\vartheta^m \varphi^{-1} \in \text{Ham}^c(M_r)$. Then $\vartheta^{mn} \varphi^{-n} \in \text{Ham}^c(M_r)$ for all $n \geq 1$. We assume without loss of generality that $r = 1$. Let W be the non-empty open subset of B constructed in 5.4, and fix $y \in W$. We first assume that $\varphi^n(D_x)$ intersects D_y transversally. Then $HF(M_1, \varphi^n(D_x), D_y)$ is defined, and in view of Proposition 5.10 and Theorem 5.14 we find that

$$\begin{aligned} \text{rank } CF(M_1, \varphi^n(D_x), D_y) &\geq \text{rank } HF(M_1, \varphi^n(D_x), D_y) \\ &= \text{rank } HF(M_1, \vartheta^{mn}(D_x), D_y) \\ &= 2mn. \end{aligned}$$

It follows that the d -dimensional submanifold $\varphi^n(D_x)$ of M_1 intersects D_y at least $2mn$ times. Since this holds true for every $y \in W$ and since $\pi: (M_1, g^*) \rightarrow (B, g)$ is a Riemannian submersion, we conclude that

$$\mu_{g^*}(\varphi^n(D_x)) \geq 2mn\mu_g(W).$$

If $\varphi^n(D_x)$ and D_y are not transverse, we choose a sequence $\varphi_i \in \text{Symp}^c(M_1)$ such that $\varphi_i^n(D_x)$ and D_y are transverse for all i , and $\varphi_i \rightarrow \varphi$ in the C^∞ -topology. For i large enough, $[\varphi_i] = [\varphi] \in \pi_0(\text{Symp}^c(M_1))$, and

$$(26) \quad \mu_{g^*}(\varphi^n(D_x)) = \lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_i^n(D_x)) \geq 2mn\mu_g(W).$$

Choose a smooth embedding $\sigma: Q^d \rightarrow T_x^*B$ such that $D_x \subset \sigma(Q^d)$. Then $\mu_{g^*}(\varphi^n(\sigma)) \geq (2m\mu_g(W))n$, and so $s_d(\varphi) \geq l(\varphi) \geq 1$, as claimed.

Assume next that $(B, g) = (\mathbb{CP}^{2n-1}/\mathbb{Z}_2, g_{\text{can}})$ and that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(M))$. Set $(\tilde{B}, \tilde{g}) = (\mathbb{CP}^{2n-1}, g_{\text{can}})$ and $\tilde{M} = T^*\tilde{B}$. In view of our normalization of g and \tilde{g} , the twist $\tilde{\vartheta}$ on \tilde{M} is a lift of ϑ , and so $\tilde{\vartheta}^m$ is a lift of ϑ^m . Lifting a symplectic isotopy between ϑ^m and φ to \tilde{M} , we obtain a symplectic isotopy between $\tilde{\vartheta}^m$ and a lift $\tilde{\varphi}$ of φ . Since the projection $\tilde{M} \rightarrow M$ is a local isometry, we thus obtain from (26) that

$$\mu_{g^*}(\varphi^n(D_x)) \geq \frac{1}{2}\mu_{\tilde{g}^*}(\tilde{\varphi}^n(D_{\tilde{x}})) \geq mn\mu_{\tilde{g}}(W)$$

for any $x \in B$ and a lift $\tilde{x} \in \tilde{B}$, so that $s_d(\varphi) \geq l(\varphi) \geq 1$.

Assume finally that (B, g) is a P -manifold modelled on S^2 different from (S^2, g_{can}) , and let ϑ be a twist defined by g . According to [45], $\pi_0(\text{Symp}^c(T^*S^2))$ is generated by the class $[\tau]$ of a generalized Dehn twist τ defined with respect to g_{can} , and so $[\vartheta] = [\tau^k]$ for some $k \in \mathbb{Z}$. Clearly, $\vartheta \neq \text{id}$. If $k = 0$, the estimate $s_1(\vartheta) = l(\vartheta) \geq 1$ therefore follows from Theorem 1, and if $k \neq 0$ from Corollary 3 below. The proof of Theorem 5.4 is complete. \square

5.6. Proof of Corollary 3. Let $\varphi \in \text{Symp}^c(T^*S^d)$ be such that $[\varphi] = [\tau^m] \in \pi_0(\text{Symp}^c(T^*S^d))$ for some $m \in \mathbb{Z} \setminus \{0\}$. Since $[\tau^2] = [\vartheta]$, we then have $[\varphi^2] = [\vartheta^m]$. Proceeding as above and assuming again $r = 1$ we find $c > 0$ such that

$$(27) \quad \mu_{g^*}(\varphi^{2n}(D_x)) \geq cn$$

for all $n \geq 1$. We denote by $\|D_z\varphi\|$ the operator norm of the differential of φ at a point $z \in T^*S^d$ with respect to g , and we abbreviate $\|D\varphi\| = \max_{z \in T^*S^d} \|D_z\varphi\|$. Using the estimate (27) we find

$$(28) \quad \begin{aligned} \mu_{g^*}(\varphi^{2n+1}(D_x)) &\geq \|D\varphi\|^{-1} \mu_{g^*}(\varphi^{2n+2}(D_x)) \\ &\geq \|D\varphi\|^{-1} c(n+1). \end{aligned}$$

The estimates (27) and (28) now show that $l(\varphi) \geq 1$, as claimed. \square

5.7. Differential topology of Dehn twists. In this subsection we collect results concerning the differential topology of Dehn twists. We shall in particular see that for odd spheres and their quotients, Theorem 3 already holds for topological reasons, while for even spheres and \mathbb{CP}^n 's, Theorem 3 is a genuinely symplectic result.

As above, (B, g) is a d -dimensional P -manifold, $M = T^*B$ and $M_r = T_r^*B$. We denote by $\text{Diff}^c(M)$ the group of compactly supported diffeomorphisms of M . Each $\varphi \in \text{Diff}^c(M)$ induces a variation homomorphism

$$\text{var}_\varphi: H_*^{cl}(M) \rightarrow H_*(M), \quad [c] \mapsto [\varphi_*c - c].$$

Here, the homology $H_*^{cl}(M)$ with closed support as well as $H_*(M)$ are taken with integer coefficients. Notice that φ is not isotopic to the identity in $\text{Diff}^c(M)$ if $\text{var}_\varphi \neq 0$. By Poincaré-Lefschetz duality,

$$H_*^{cl}(M) \cong H_*^{cl}(M_r \setminus \partial M_r) \cong H_*(M_r, \partial M_r) \cong H^{2d-*}(M_r) \cong H^{2d-*}(B),$$

and $H_*(M) \cong H_*(B)$, and so $\text{var}_\varphi = 0$ except possibly in degree $* = d$. It is known from classical Picard-Lefschetz theory that $\text{var}_\tau: H_d^{cl}(T^*S^d) \rightarrow H_d(T^*S^d)$ does not vanish, see [4, p. 26]. Assume now that B is oriented. We orient the fibres T_x^*B , $x \in B$, such that $[B] \cdot [T_x^*B] = 1$, where \cdot denotes the intersection product in homology determined by the natural orientation of the cotangent bundle M . Then $H_d^{cl}(M) \cong \mathbb{Z}$ is generated by the fibre class $F = [T_x^*B]$, and $H_d(M) \cong \mathbb{Z}$ is generated by the base class $[B]$, which by abuse of notation is denoted B .

Proposition 5.20. *Assume that (B, g) is an oriented P_k -manifold.*

- (i) *If k is even and d is odd, $\text{var}_{\vartheta^m}(F) = 2mB$ for $m \in \mathbb{Z}$.*
- (ii) *If k is odd, $\text{var}_{\vartheta^m} = 0$ for all $m \in \mathbb{Z}$.*

Proof. For simplicity we assume again $m \geq 1$. As in Subsection 5.4 we choose $\vartheta = \vartheta_f$, fix $x \in B$, choose $y \in W$, and let $\vartheta^m(T_x^*B) \cap T_y^*B = \{c_0^\pm, \dots, c_{m-1}^\pm\}$. The local intersection number of $\vartheta^m(T_x^*B)$ and T_y^*B at c_i^\pm is $(-1)^{\text{ind}(c_i^\pm)}$. Recall from Proposition 5.11 that $\text{ind } c_0^+ = 0$ and

$$\text{ind } c_i^- = \text{ind } c_i^+ + k \quad \text{and} \quad \text{ind } c_{i+1}^+ = \text{ind } c_i^+ + k + d - 1.$$

- (i) If k is even and d is odd, we find

$$\vartheta_*^m(F) \cdot F = \sum_{i=0}^{m-1} (-1)^{\text{ind } c_i^+} + (-1)^{\text{ind } c_i^-} = \sum_{i=0}^{m-1} 2 = 2m,$$

and so $\text{var}_{\vartheta^m}(F) \cdot F = \vartheta_*^m(F) \cdot F - F \cdot F = 2m$, i.e., $\text{var}_{\vartheta^m}(F) = 2mB$.

- (ii) If k is odd, we find

$$\vartheta_*^m(F) \cdot F = \sum_{i=0}^{m-1} (-1)^{\text{ind } c_i^+} + (-1)^{\text{ind } c_i^-} = 0,$$

and so $\text{var}_{\vartheta^m}(F) \cdot F = 0$, i.e., $\text{var}_{\vartheta^m}(F) = 0$. □

Before discussing the variation homomorphism further, let us show how Proposition 5.20 (i) leads to a topological proof of Theorem 3 for the known odd-dimensional P -manifolds.

Corollary 5.21. *Assume that (B, g) is a round sphere S^{2n+1} or one of its quotients S^{2n+1}/G or a Zoll manifold (S^{2n+1}, g) . Then the conclusion of Theorem 3 holds true. Moreover, if $\varphi \in \text{Diff}^c(T^*B)$ is such that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Diff}^c(T^*B))$ for some $m \in \mathbb{Z} \setminus \{0\}$, then $s_d(\varphi) \geq 1$.*

Proof. According to Proposition 5.1, (B, g) is a P_0 -manifold or a P_{2n} -manifold, and so Proposition 5.20 (i) shows that $\varphi_*^n(F) = 2mnB + F$ for all $n \geq 1$. Choose $r < \infty$ so large that φ is supported in T_r^*B , choose $x \in B$ and set $D_x(r) = T_x^*B \cap T_r^*B$. Then

$$\mu_{g^*}(\varphi^n(D_x(r))) \geq (2m\mu_g(B))n$$

for all $n \geq 1$, and so the corollary follows. \square

We now turn to the known even-dimensional P -manifolds. Propositions 5.1 and 5.20 (ii) show that $\text{var}_{\vartheta^m} = 0$ for S^{2n} , \mathbb{CP}^n , \mathbb{HP}^n , \mathbb{CaP}^2 and even-dimensional Zoll manifolds and all $m \in \mathbb{Z}$. For the non-orientable spaces \mathbb{RP}^{2n} and $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$ the vanishing of var_{ϑ^m} follows from $H_{2n}(\mathbb{RP}^{2n}) = 0$ and $H_{4n-2}(\mathbb{CP}^{2n-1}/\mathbb{Z}_2) = 0$. The variation homomorphism can be defined for homology with coefficients in any Abelian group G , and one checks that var_{ϑ^m} vanishes over any finitely generated G for all the above even-dimensional P -manifolds and every $m \in \mathbb{Z}$. Note that if $\text{var}_{\vartheta^m} \neq 0$ for some $m \neq 0$ then ϑ is not isotopic to the identity in $\text{Diff}^c(M)$. Since we are not aware of another obstruction we ask

Question 5.22. Assume that (B, g) is one of \mathbb{RP}^{2n} , \mathbb{HP}^n , \mathbb{CaP}^2 , $\mathbb{CP}^{2n-1}/\mathbb{Z}_2$. Is ϑ isotopic to the identity in $\text{Diff}^c(T^*B)$?

We did not ask Question 5.22 for even-dimensional Zoll manifolds or \mathbb{CP}^n in view of

Proposition 5.23. *Assume that (B, g) is an even-dimensional Zoll manifold or \mathbb{CP}^n . Then ϑ is isotopic to the identity in $\text{Diff}^c(T^*B)$.*

Proof. This has been proved in [47] for $B = \mathbb{CP}^n$, $n \geq 1$, by extending the construction for S^2 given in [46]. This construction carries over literally to S^6 since S^6 carries an almost complex structure induced by the vector product on \mathbb{R}^7 related to the Cayley numbers, see [33, Example 4.4]. For arbitrary even-dimensional spheres the result is proved by N. Krylov, [32]. Finally, let $\vartheta = \vartheta_f$ be a twist defined by a Zoll metric g on S^{2n} . Then there is a smooth family g_t , $t \in [0, 1]$, of P -metrics

with $g_0 = g_{\text{can}}$ and $g_1 = g$. The family ϑ_t of twists defined by f and g_t is then a smooth family in $\text{Symp}^c(T^*S^{2n})$. In particular, $\vartheta = \vartheta_1$ is isotopic to ϑ_0 and hence to the identity in $\text{Diff}^c(T^*S^{2n})$. \square

Remark 5.24. One can show that ϑ is isotopic to the identity in $\text{Diff}^c(T^*B)$ for any *almost complex* P -manifold B . The only almost complex manifolds among the known P -manifolds are, however, \mathbb{CP}^n , $n \geq 1$, and S^6 .

6. REMARKS ON SMOOTHNESS

Entropy type estimates often depend on the differentiability of the maps considered. E.g., in the case of finite smoothness the entropy conjecture (4) has been proved only for special classes of manifolds and maps, and there exist homeomorphisms φ of compact manifolds such that $h_{\text{top}}(\varphi) < \log \rho(\varphi_*)$, see [28]. The results established in this paper hold under essentially minimal differentiability assumptions necessary to formulate them. This is so because the uniform lower bounds found for $s_1(\varphi)$, $s_d(\varphi)$ and $l(\varphi)$ and C^∞ -smooth symplectomorphisms are of “symplecto-topological” nature. Given a symplectic manifold (M, ω) , let $\text{Symp}^{c,1}(M, \omega)$ be the group of C^1 -smooth symplectomorphisms whose support is compact and contained in $M \setminus \partial M$, and let $\text{Symp}_0^{c,1}(M, \omega)$ be its identity component.

Proposition 6.1. *Theorem 3.1 holds true for $\varphi \in \text{Symp}_0^{c,1}(M, d\lambda)$.*

Proof. Proposition 3.2 is proved in [20] for C^2 -smooth Hamiltonians. The Flux is defined on $\text{Symp}_0^{c,1}(M, \omega)$, and the exact sequence (6) exists in the C^1 -category. The remaining arguments in the proof of Theorem 3.1 are of topological nature and thus go through for C^1 -smooth symplectomorphisms. \square

Corollary 3.6, Theorem 3.7 and Corollary 3.9 also continue to hold in the C^1 -category.

Proposition 6.2. *Theorem 2 holds true for C^2 -smooth classical Hamiltonian functions.*

Proof. The results of [5] hold for C^2 -smooth Hamiltonians, and the arguments in Section 4 go through. \square

We endow $\text{Symp}^{c,1}(M, \omega)$ with the C^1 -topology. According to a result of Zehnder, [62], $\text{Symp}^c(M, \omega)$ is dense in $\text{Symp}^{c,1}(M, \omega)$, and by a result of Weinstein, [33, Theorem 10.1], both groups are locally path

connected. It follows that the inclusion $\text{Symp}^c(M, \omega) \rightarrow \text{Symp}^{c,1}(M, \omega)$ induces an isomorphism of mapping class groups, $\pi_0(\text{Symp}^c(M, \omega)) = \pi_0(\text{Symp}^{c,1}(M, \omega))$.

Proposition 6.3. *Theorem 5.4 and Corollary 3 hold true for C^1 -smooth symplectomorphisms.*

Proof. Let (B, g) be as in Theorem 5.4, and let $\varphi \in \text{Symp}^c(T^*B)$ be a C^1 -smooth symplectomorphism such that $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(T^*B))$ for some $m \in \mathbb{Z} \setminus \{0\}$. We can assume that φ is supported in T_1^*B . Choose a sequence $\varphi_i \in \text{Symp}^c(T_1^*B)$ of C^∞ -smooth symplectomorphisms such that $\varphi_i \rightarrow \varphi$ in the C^1 -topology. For i large enough, $[\varphi_i] = [\varphi] \in \pi_0(\text{Symp}^c(T^*B))$. Using the estimate (26) we thus conclude

$$\mu_{g^*}(\varphi^n(D_x)) = \lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_i^n(D_x)) \geq 2mn\mu_g(U)$$

for all $n \geq 1$. Therefore, $s_d(\varphi) \geq l(\varphi) \geq 1$. Corollary 3 now follows also for C^1 -smooth symplectomorphisms. \square

7. COMPARISON OF VOLUME GROWTH AND GROWTH OF THE DIFFERENTIAL

For any compactly supported diffeomorphism φ of a smooth manifold M we denote by $\|D_x\varphi\|$ the operator norm of the differential of φ at the point x with respect to some Riemannian metric on M . Following [11] we define the *growth sequence* of φ by

$$\Gamma_n(\varphi) = \max_{x \in M} \|D_x\varphi^n\|.$$

In [39], Polterovich proved that if (M, ω) is a closed symplectic manifold with $\pi_2(M) = 0$, then for a large class of symplectomorphisms $\varphi \in \text{Symp}_0(M, \omega)$ there exists a uniform lower bound for the growth type of the sequence $\Gamma_n(\varphi)$. Complementary results for symplectic and smooth diffeomorphisms were found in [40] and [41]. Here, we derive from the growth sequence the *slow differential growth*

$$\gamma(\varphi) = \liminf_{n \rightarrow \infty} \frac{\log \Gamma_n(\varphi)}{\log n}.$$

It does not depend on the choice of Riemannian metric. While in the definition of the slow volume growth $s_i(\varphi)$ we looked at the weighted asymptotics of the most distorted smooth i -dimensional family of orbits, in the definition of $\gamma(\varphi)$ we look at each time n at the place of the strongest distortion and pass to a weighted limit. The invariants $s_i(\varphi)$

are thus of rather dynamical nature, while $\gamma(\varphi)$ is of rather geometric nature. Clearly,

$$(29) \quad s_i(\varphi) \leq i \gamma(\varphi), \quad i = 1, \dots, \dim M.$$

We therefore read off from our main results

Corollary 7.1. *Under the assumptions of Theorem 3.1 it holds that $\gamma(\varphi) \geq 1$.*

Corollary 7.2. *Under the assumptions of Theorem 2 it holds that $\gamma(\varphi) \geq 1/2$ respectively $\gamma(\varphi) \geq 1$.*

Corollary 7.3. *Under the assumptions of Theorem 5.4 or Corollary 3, it holds that $\gamma(\varphi) \geq 1/d$.*

Our proof of Proposition 5.3 (ii) shows that $\gamma(\vartheta) = 1$ for any twist ϑ of the cotangent T^*B over a P -manifold (B, g) . It follows that $\gamma(\tau) = 1$ for any (generalized) Dehn twist τ of T^*S^d .

Question 7.4. Can the lower bound $1/d$ in Corollary 7.3 be replaced by 1?

The estimates (29) are in general not sharp: Choose a monotone decreasing smooth function $f: \mathbb{R} \rightarrow [1, 2]$ with $f(r) = 2$ if $r \leq 1$ and $f(r) = 1$ if $r \geq 2$, and define $\phi \in \text{Diff}_0^c(\mathbb{R}^n)$ by $\phi(x) = f(|x|)x$. Using ϕ as a plug, we see that every smooth manifold M carries $\varphi \in \text{Diff}_0^c(M)$ with $s(\varphi) = 0$ and $\gamma(\varphi) = \infty$. It would be interesting to find such diffeomorphisms in the symplectic category.

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